Lecture 10

- Work
- Gravitational Potential Energy
- Power

Cutnell+Johnson: 6.3-6.7, 5.7

Let's review what we did last time a bit. We saw last time that there were two ways to define total work. First of all, the total work is the change in kinetic energy:

$$W = \Delta KE$$

We saw that this is equivalent to defining

$$W = (\Sigma F) d \cos(\theta)$$

where θ is the angle between the net force and the displacement. The latter form means that if the a net force is applied over a distance, work is being done on the object.

A few comments:

1. Work can be positive or negative. Negative work on an object means it is slowing down, i.e. losing kinetic energy.
2. Work and energy are not vectors; they have no direction associated with them.
3. For those who remember vector analysis, another way of writing the formula for work is as

   $$W = \vec{F} \cdot \vec{d}$$

   This is called the dot product of two vectors, and is $\vec{F} \cdot \vec{d} = Fd \cos\theta$. It tells you how to get something which is not a vector (the work) from two vectors (the force and the displacement). In components, the dot product of two vectors $\vec{A} = (A_x, A_y)$ and $\vec{B} = (B_x, B_y)$ is

   $$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$

Instead of just dealing with net force and total work, we can study work done by different forces in a problem. The same kind of formula applies: if I apply a force F_{me} over a distance d, with θ the angle between the displacement and the force I apply, then I do

$$W_{me} = F_{me} d \cos(\theta)$$

If you just then add up all the work done by all the forces in the problem, then you end up with the formulas above.

Problem

I sweep the ice with a $1\,kg$ broom, using a force of $20N$ at an angle of 45^o with the horizontal. How far do I have to push it to give it a speed of $4\,m/s$, if the coefficient of kinetic friction $\mu_k = .3$? How much work have I done? How much work has friction done?

Answer

To compute the net force, we draw a picture. We have

$$\sum F_x = F_{me}\cos(45^o) - F_{fr}$$

$$\sum F_y = F_N - F_{me}\sin(45^o) - mg = 0$$

Since the sum of the forces in the y direction must be zero, this lets us find F_N and then F_{fr}:

$$F_{fr} = \mu_k F_N = \mu_k(F_{me}\sin(45^o) + mg) = (.3)\left((20\,N)(\sqrt{2}/2) + (1\,kg)(9.8\,m/s^2)\right) = 7.2\,N$$

This means that the sum of the forces in the x direction is

$$\sum F_x = (20\,N)(\cos(45^o)) - 7.2\,N = 7.0\,N$$

Since the displacement in the x direction, the total work done is therefore

$$W = (\sum F_x)d$$

We can determine the total work another way: the change in kinetic energy of the broom is

$$\Delta KE = \frac{1}{2}mv^2 = \frac{1}{2}1\,kg(4m/s)^2 = 8J$$

Thus

$$d = \frac{W}{\sum F_x} = \frac{8\,J}{7.0\,N} = 1.1\,m$$

To get the work done by me, use

$$W_{me} = F_{me} d\cos(45^o) = (20\,N)(1.1\,m)\frac{1}{\sqrt{2}} = 16.2J$$

To get the work done by friction, use

$$W_{fr} = F_{fr} d\cos(\theta)$$

Notice here that $\theta = 180^o$, because friction is opposing the motion. Because $\cos(180^o) = -1$, we have

$$W_{fr} = (7.0\,N)(1.1\,m)(-1) = -8.1\,J$$

The work done by friction is always negative. Note the two works add up (up to rounding error) to give the total work:

$$W = W_{fr} + W_{me} = 8J$$

The normal does no work, because it is perpendicular to the motion.

Gravitational Potential Energy

Energy can be a little subtle. Say I lift up a block, gradually slowing down to a stop. Thus the net change in kinetic energy is zero, but certainly I had to use energy to lift it up. In other words, I did work: I applied a force over a distance to get it there. Where did this energy go? Moving it slowly means that there is very little friction, so it isn't going to heat. Of course, the energy can be converted back to kinetic energy by dropping it. So where does the energy go in between?

The answer is that the energy I expended lifting the block turns into *gravitational potential energy*. Think about what happens if I throw a ball straight up into the air (neglecting friction) and catch it when it comes down. It has an initial kinetic energy. As the ball goes up, it gets slower, thus losing kinetic energy. This kinetic energy is converted to potential energy as it gets higher. When it reaches its peak, all of the kinetic energy has been converted to potential energy. Then as it falls, it loses potential energy, but gains kinetic energy. You'll remember from earlier lectures that the speed when it hits my hand is the same as the speed as when it had left (the velocity of course has changed sign). Thus the kinetic energy when I catch it is the same as when I threw it. Since the final position of the ball is the same as its initial position, its potential energy is the same at the end as at the beginning.

The point of the last paragraph can be summarized in one simple formula. In the absence of forces other than gravity, then

$$\Delta KE + \Delta PE = 0$$

This is one application of the principle of conservation of energy. Energy is neither being created not destroyed: it is just converted between kinetic and potential.

To make the concept of gravitational potential energy useful, we need an explicit formula. Let's derive one, using our example of throwing a ball up in the air. Say I throw a ball of mass m vertically, with an initial speed v which rises to a height h. You could compute h by using our formulas for velocity, acceleration, distance, etc. Let's do it using the formulas for work we introduced last time. The force slowing down the ball is gravity, which is of magnitude mg. This is the only force in the problem (once you've let go of the ball). The displacement is up, while the force is down, so the angle between the force and the displacement is 180^o (i.e. they are opposite). Thus the work done from when I throw the ball until its peak is

$$W = Fd\cos\theta = mgh\cos(180^o) = -mgh$$

This work must be the change in kinetic energy. This is enough information to get the change in potential energy:

$$\Delta PE = -\Delta KE = -W = mgh$$

Thus this object raised to a height h has its potential energy increased by mgh.

Notice that here the potential energy is independent of the initial velocity v. In fact, the gravitational potential energy is independent of how you get that object to that height (e.g. it I lift it slowly or throw it up). Thus the general formula for gravitational potential energy is

$$PE = mgh$$

It doesn't matter where you set the zero of height, because the only thing ever necessary in these problems is the change in potential energy ΔPE. Thus only the change in height matters, so where you set $h = 0$ doesn't matter (you just may have to deal with negative heights if you're interested in positions below where you set the zero).

This formula is very useful. For example, it lets us compute the distance h in the above example in terms of g easily. We have

$$\Delta PE = mgh$$

Since the final kinetic energy is zero, we have

$$\Delta KE = 0 - \frac{1}{2}mv^2$$

Since there are no forces other than gravity,

$$\Delta KE = -\Delta PE$$

This gives

$$h = \frac{v^2}{2g}$$

One simple application of potential energy is that if you drop a ball, it can only bounce back to its original height (in the absence of other forces). If there is a force like friction, then this converts some of the potential energy to heat. This means the ball has to return to less than its original height. Similarly, a bowling-ball pendulum can only return to its initial position, no further.

Let's do a more complicated problem, which combines these ideas with the work we did last section on centripetal acceleration.

Problem A ramp is attached to a loop de loop of radius r. How high on the ramp does the ball need to start in order to successfully navigate the loop?

Answer Note I did not ask how high the ball had to be in order to merely reach the top of the ramp. Just to reach the top, it needs to be just as high as the ramp. However, if it were only the high, it would just make it to the top, then plummet to the ground. In order to complete the circle, it needs to have enough velocity so that it can complete its centripetal motion. In other words, it must be going fast enough to make it all the way around. So first, let's figure out how fast this must be. At the top of the loop, the force of gravity is of course mg pushing down. There is a normal force from the loop also pushing downward. Thus the total force is of magnitude

$$F_N + mg$$

For the object to be moving in a circle, this must be equal to mv^2/r, directed towards the center, so for it to be moving in a circle, we have

$$F_N + mg = m\frac{v^2}{r}$$

I emphasize that this equation must be satisfied if the ball is to move in a circle; we derived this equation in lecture 12 in the yo-yo problem. We want the minimum velocity so that this is possible. The key is noticing that that F_N always must be positive. So the smallest velocity which allows the ball to be moving occurs when $F_N = 0$. This should be clear to you – this corresponds to the case where the ball just barely makes it around. So we have

$$mg = m\frac{v_{top}^2}{r}$$

Again, by v_{top} I mean the minimum velocity for the ball not to fall out of the loop. Thus the kinetic energy at the top of the loop must be

$$KE_{top} = \frac{1}{2}mv_{top}^2 = \frac{1}{2}mgr$$

To ensure that the ball has enough kinetic energy to make it around, the ball must be placed high enough on the ramp. It is easy to calculate how high using our formula for potential energy. Starting from rest, the desired change in kinetic energy is

$$\Delta KE = KE_{top}$$

This must be equal to minus the change in potential energy, which in turn must be $-mg\Delta h$. In a formula, we have

$$\Delta PE = mg\Delta h = -\Delta KE = \frac{1}{2}mgr$$

Thus the change in height of the ball is

$$\Delta h = -\frac{r}{2}$$

In other words, the ball must be placed a distance $r/2$ higher than the *top* of the loop. Δh is negative because the height is decreasing: the final height is lower than the initial.

One of the reasons it is much easier to do these problems with energy instead of detailed force calculations is that you don't need to worry about the angle the ball is moving at. Note that this result is independent of how the ramp is tilted. These energy computations let you ignore that: if I ball is some height h lower than it was originally, then it must be at a speed $v = \sqrt{2gh}$. This is why requiring the conservation of energy is so powerful.

Of course, real life is more complicated. First of all, there is friction, which converts some of the potential energy to heat instead of kinetic energy. Thus in the loop-de-loop problem, you must place the ball higher than $r/2$ to compensate for this. In the loop-de-loop problem another effect we have ignored is the rotation of the ball. This is another form of kinetic energy, which we will discuss later in the course. Thus the ball has to go a little higher up the ramp in order to give it its proper rotational energy.

Friction is an example of what is called a *non-conservative* force. Roughly speaking, a non-conservative force is one which you "can't get back". Gravity is a conservative force: what goes up must come down (and with the same velocity). Once you've converted the energy to heat, you can't convert it easily back to kinetic or potential (we'll discuss this issue when we discuss heat later in the course). Another example of a non-conservative force is a jet engine: you can't put back all the air it spewed out the back. Conservative forces have potential energies associated with them. Gravity isn't the only one. For example, if you push a spring down, you also end up with potential energy: the moment you let go of the spring, it pushes back. Many materials have an elastic energy: if you bend them, they want to bend back.

Power

Power is quite simply the rate at which you are doing work. In other words, if you do an amount of work W in some time Δt, then the average power $\overline{P}$ is

$$\overline{P} = \frac{W}{\Delta t}$$

Equivalently, the power is

$$\overline{P} = \frac{\Delta KE}{\Delta t}$$

The MKS unit of power is a joule per second, which gets its own name, the watt, abbreviated by W (not to be confused with the weight!). Thus

$$1W = 1J/s = 1kgm^2/s^3$$

Notice that the way your power company bills you is in kilowatt hours. This is an energy: a kWh is the amount of energy which is used when you use $1kW = 1000W$ for one hour. Notice how the units work out.