Lecture 11

- Impulse/Momentum
- Conservation of Momentum

Cutnell+Johnson: 7.1-7.3

Impulse and Momentum

We learned about work, which is the force times distance (times the cosine of the angle in between the two). The longer the distance you apply the force over, the more work you do. The *impulse* is defined similarly. For a constant force $\vec{F}$ applied over a time interval Δt, the impulse is

$$\vec{F}\Delta t$$

As opposed to work, impulse is a vector: it has a direction. As with work, F is the net force: like last time, the book and I omit the Σ.

You probably haven't heard of the impulse before. The reason is that it's closely related to an important quantity, the momentum, so people usually talk about momentum instead. Remember that work is the change in the kinetic energy. The impulse is the change in momentum $\vec{p}$, namely

$$\Delta\vec{p} = \vec{F}\Delta t$$

We can write out another formula for the momentum using Newton's second law. This law gives

$$\vec{F} = m\vec{a}$$

To get the momentum, multiply both sides by Δt :

$$\Delta\vec{p} = \vec{F}\Delta t = m\vec{a}\Delta t$$

We can now use the definition of acceleration:

$$\vec{a} = \frac{\Delta\vec{v}}{\Delta t}$$

to give

$$\Delta\vec{p} = m\Delta\vec{v}$$

Thus the change in momentum is the mass times the change in velocity, and if we define zero momentum to be zero velocity, we end up finally with a very useful formula for momentum:

$$\vec{p} = m\vec{v}$$

So as with work, using momentum can be a convenient way to solve some problems.

Problem A 2000 kg car crashes into a wall at 10 mph (=4.5 m/s). It comes to a stop in .1 sec. What average force is applied to bumper? (The bumper is the only part of the car which hits the wall.)

Answer The initial momentum of the car is of magnitude

$$p = mv = (2000\,kg)(4.5\,m/s) = 9000kg\,m/s$$

Notice the unit of momentum is $kg\,m/s = N\,s$. It doesn't get its own name like energy or force (feel free to call it the Fendley). This momentum changes from this initial to zero in $.1sec$. Thus

$$\Delta p = 0 - 9000kg\,m/s = -9000kg\,m/s$$

This is equal to force times Δt, so

$$F = \frac{\Delta p}{\Delta t} = \frac{-9\times 10^3 kg\,m/s}{.1s} = -9\times 10^4 N$$

That's an enormous force for the bumper to withstand. The negative sign means that the force is directed opposite to the direction of the momentum, which is of course because the force is slowing the car down.

Conservation of Momentum

In the last few lectures and in this lecture so far, we've done a variety of problems involving work, energy and momentum. However, all of these problems could have been done without knowing anything about work or momentum: the methods of the earlier chapters were sufficient (although maybe a little more complicated) to solve the problem. We did introduce one new principle, the conservation of energy. However, for conservative forces like gravity, the conservation of energy already is implied by the kinematic equations we were using (e.g. $d = v_0t + at^2/2$.)

I'm going to introduce here something which is not implied by anything we've done earlier. Because this is a new fact, it will prove very powerful, so powerful we call it a law. This is the law of the *conservation of momentum*:

The *total* momentum of a system does not change *if there are no external forces.*

Just like energy, momentum can be transferred from object to object. This is what a force does; recall that force is $\Delta p/\Delta t$. What is conserved is overall momentum. That means that when objects collide, their individual momenta may increase or decrease, but the total momentum must stay the same.

So let's consider a collision between two objects. If the first object (mass m_1) has an initial velocity $\vec{v}_{1i}$ and the second (mass m_2) an initial velocity $\vec{v}_{2i}$, their initial total momentum is

$$\vec{p}_i = m_1\vec{v}_{1i} + m_2\vec{v}_{2i}$$

Note that this is a vector sum: you need to add the components individually. After the collision, they move off in new directions with final velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$. The total final momentum is therefore

$$\vec{p}_f = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

When the objects collide with each other, there are certainly forces. Thus the individual momenta are not conserved. However, if we consider the system of both objects together, there are no external forces: the only forces are those of one block on the other and the reaction. Thus the total momentum of the two blocks is conserved. Momentum conservation means that

$$\vec{p}_i = \vec{p}_f$$

or equivalently

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = m_1\vec{v}_{1f} + m_2\vec{v}_{2f}$$

The fact that this is a vector equation means that it holds true for any component. For example, if $\vec{v}_{1i} = (v_{1ix}, v_{1iy})$ etc.,

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

and likewise for the y component. Notice that given the initial velocities, this is not enough information to determine the final velocities uniquely. There is one equation for each component, but two unknowns.

Problem A 1000 kg car moving north at 30 mph ($13.4m/s$) collides with a 2000 kg SUV moving east at 50 mph ($22.4m/s$). The cars get entangled and move off together. What is their final speed? How much energy is converted to other forms?

Answer Say we define our coordinates so that north is the y direction and east is the x direction. Then the initial momentum of the first car is

$$\vec{p}_1 = (0, (1000\,kg)(13.4\,m/s)) = (0, 1.34 \times 10^4\,kg\,m/s)$$

while the momentum of the second is

$$\vec{p}_2 = ((2000\,kg)(22.4\,m/s), 0) = (4.48 \times 10^4\,kg\,m/s, 0)$$

The final momentum must be the same as the initial momentum:

$$\vec{p}_{final} = \vec{p}_{initial} = (4.48 \times 10^4\,kg\,m/s, 1.34 \times 10^4\,kg\,m/s)$$

Since both cars end up tangled together, they move at the same velocity $\vec{v_f}$. The final momentum therefore is

$$\vec{p}_{final} = (m_1 + m_2)\vec{v}_f = ((3000\,kg)v_{fx}, (3000\,kg)v_{fy})$$

Therefore the x component is

$$v_{fx} = \frac{4.48 \times 10^4\,kg\,m/s}{3000\,kg} = 14.9 m/s$$

$$v_{fy} = \frac{1.34 \times 10^4\,kg\,m/s}{3000\,kg} = 4.47 m/s$$

Thus the cars together are moving east-north-east (closer to east than north). Their speed is

$$v_f = \sqrt{v_{fx}^2 + v_{fy}^2} = \sqrt{(14.9)^2 + (4.47)^2} m/s = 15.6 m/s$$

As for how much energy is converted to other forms, the initial kinetic energy is

$$\begin{aligned} KE_i &= \frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 \\ &= \frac{1}{2}(1000)(13.4)^2 + \frac{1}{2}(2000)(22.4)^2 = 5.91 \times 10^5 J \end{aligned}$$

while the final kinetic energy is

$$KE_f = \frac{1}{2}(3000)(15.6)^2 J = 3.65 \times 10^5 J$$

So over a third of the initial kinetic energy is converted to other forms of energy (the energy to crumple the bumper, heat, etc).

In many situations, though, you do have enough information to find the velocities. Here I'll show how to use the conservation of momentum and energy to measure the speed of a bullet. This is done with what is called a *ballistic pendulum*. The idea is simple. You fire a bullet into a pendulum, which swings upward. By measuring how high the pendulum swings, and using the conservation of energy and momentum, you can deduce the initial momentum and velocity.

We divide the computation into two stages. First, let's study the collision between the bullet (mass m_b) and the pendulum (mass m_P). In this collision, energy goes for example into heat (the friction which stops the bullet) and chemical energy (breaking apart parts of the pendulum so the bullet can lodge inside). Thus kinetic energy is not conserved in this collision. However, in the collision, there are no external forces. No external forces means that the only forces are the bullet hitting the pendulum, and the reaction force of the pendulum. These are internal forces. Since there are no external forces, the total momentum of the bullet and pendulum is conserved. Before the collision, the velocity of the pendulum is zero while the speed of the bullet is v_b. Thus the initial momentum of the pendulum is zero, and the total initial momentum (the momentum before the collision) is

$$p_{before} = m_b v_b$$

The direction of the momentum is of course in the direction of the bullet's motion. In the collision, the bullet lodges in the pendulum. Just after the collision, the bullet and the pendulum move with a speed v_a (the subscript is for combined) Thus the momentum just after the collision is

$$p_{after} = (m_b + m_P)v_a$$

By conservation of momentum, the momentum does not change in the collision:

$$p_{before} = p_{after}$$

Thus

$$m_b v_b = (m_b + m_P)v_a$$

We can solve this for the initial speed of the bullet:

$$v_b = \frac{m_b + m_P}{m_b} v_a$$

So if we know the velocity of the pendulum, we can figure out the initial velocity of the bullet.

To figure out the velocity of the pendulum, we use conservation of energy. The pendulum swings up to a height h, which we can easily measure. We relate h to the velocity simply by using the conservation of energy. After the collision, the force of gravity is the only force acting on the system. Since gravity is an external force, momentum is not conserved. However, energy is conserved (see why gravity is called a conservative force?). The kinetic energy just after the collision is $(m_b + m_P)(v_a)^2/2$, while when the pendulum is at its peak the kinetic energy is zero. Thus

$$\Delta KE = -\frac{1}{2}(m_b + m_P)(v_a)^2$$

The change in height is h, so the change in gravitational potential energy is

$$\Delta PE = (m_P + m_b)gh$$

Now we can use our equation

$$\Delta KE + \Delta PE = 0$$

to give

$$\frac{1}{2}(v_a)^2 = gh$$

Solving for the velocity gives

$$v_a = \sqrt{2gh}$$

We derived this in the last lecture, in fact. It wasn't for a pendulum, but one of the points I made last time (and I'll make it again here) is that it doesn't matter what sort of system you have: energy is conserved (the tension in the string does no work because its force is perpendicular to the displacement). Anyway, conservation of momentum relates the bullet's velocity to the pendulum's velocity after the collision, and conservation of energy relates this to the height to which the pendulum swings. Putting these two together gives finally

$$v_b = \frac{m_b + m_P}{m_b} v_a = \frac{m_b + m_P}{m_b} \sqrt{2gh}$$