Lecture 12

- More on Conservation of Momentum
- Elastic Collisions

Cutnell+Johnson: 7.2-7.5

Conservation of Momentum

The conservation of momentum is a quite powerful tool. Newton's first law is just a special case applying to a single object: if there are no forces, the object continues to move at the same velocity. When there are multiple objects, the total momentum must stay the same, as long as there are no external forces. The reason internal forces don't change the momentum is that if object 1 pushes on object 2 with some force $\vec{F}$, then object 2 must push back on object 1 with the opposite force $-\vec{F}$. Thus the net force on the *whole* system of 1 and 2 together is $\Sigma\vec{F} = \vec{F} + (-\vec{F}) = 0$. Remember that

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$$

Since $\vec{F} = 0$ here, the momentum does not change.

This is true even if kinetic energy is not conserved. You can easily check in the ballistic pendulum problem that the initial kinetic energy of the bullet before the collision is much greater then that of the combined bullet-pendulum system. Nevertheless, the momentum of the bullet-pendulum system just after the collision is the same as before the collision. Of course, overall energy is conserved, but kinetic energies may be conserved.

The fact that momentum can be conserved while kinetic energy is not is extremely obvious in this example.

Problem Two blocks of mass m_1 and m_2 are at rest, with a compressed spring (of negligible mass) in between. The spring uncompresses, and sends the blocks moving. What is the ratio of speeds of the two blocks?

Answer Before the spring uncompresses, the blocks are not moving, so the initial total momentum is zero. The spring uncompresses and pushes the blocks. This is a force entirely within the spring-block system, so there are no external forces. Therefore, the momentum after the spring uncompresses must remain zero. Thus

$$\vec{p}_1 + \vec{p}_2 = 0$$

or in other words, the final momentum of the first block is the opposite of the second block. Notice that both components of the total momentum must vanish. This means that the blocks must go in opposite directions. Otherwise, there's no way that both components of the momentum would vanish. So let's call the direction block 1 moves the positive x direction, and the direction block 2 moves the negative x direction. Then

$$p_{1x} = mv_{1x} \qquad p_{1y} = 0$$

$$p_{2x} = mv_{2x} \qquad p_{2y} = 0$$

Now we can use the conservation of momentum to relate the two momentum:

$$\begin{aligned} p_{1x} + p_{2x} &= 0 \\ m_1 v_{1x} + m_2 v_{2x} &= 0 \end{aligned}$$

Thus the ratio of speeds is

$$\frac{v_1}{v_2} = \frac{m_2}{m_1}$$

As you would expect, the heavier object moves slower: An object 3 times as heavy moves at one-third the speed.

Elastic Collisions

As I've emphasized, kinetic energy need not be conserved. However, sometimes it is. This is called an *elastic* collision. The name is a good one. One example of an elastic collision is when a ball bounces off a wall: it comes out with the same speed and hence the same kinetic energy as when it went in.

Problem A ball of mass $.25\,kg$ and speed $5m/s$ collides head-on with a ball of mass $.75\,kg$ which is at rest. If the collision is elastic, what are the final velocities?

Answer There are no external forces, so momentum is conserved. Since the collision is head on, there are no forces (internal or external) which cause the balls move out of a one-dimensional line. Thus this is a one-dimensional problem. Conservation of momentum says that

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Because the collision is elastic, kinetic energy is conserved as well, so

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

We now have two equations, two unknowns. The equations simplify a bit because $v_{2i} = 0$. Thus we solve for say v_{2f} in the first equation, and plug it into the second. After a little algebra, you end up with

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$
$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

Plugging in the numbers gives $v_{f1} = -2.5m/s$ and $v_{f2} = 2.5m/s$. The minus sign means that the first block bounces backwards. It is a special fluke of having the stationary block 3 times the mass of the moving block, that the two final speeds are equal. For example, say they were the same mass. Then $v_{1f} = 0$ and $v_{2f} = v_{1i}$.

Problem A little basketball (mass m) is placed on top of a big one (mass M). The two are dropped from a height h. How high does the little one bounce? Assume the collisions are elastic, and that M is much larger than m.

Answer First think about the big ball. Since it is at a height h, we can use conservation of energy to find its velocity right before it hits the ground. We have

$$Mgh = \frac{1}{2}MV^2$$

so

$$V_{before} = \sqrt{2gh}$$

When it hits the ground, it bounces upward. Since the collision is elastic, it bounces upward with the same speed it had initially, $V_{before} = \sqrt{2gh}$. Now let's think about the little ball. Instead of hitting the ground, it hits the big ball going up. Its initial velocity is

$$mgh = \frac{1}{2}Mv^2$$

so v before it hits the big ball is also $v_{before} = V_{before} = \sqrt{2gh}$. Now this is the hard part. We have an elastic collision with the big ball moving **upward** with speed $v_{before} = \sqrt{2gh}$, and the little ball moving **downward** with speed $v_{before} = \sqrt{2gh}$. We can now use conservation of momentum and energy to find the velocities V_{after} and v_{after}. The conservation of momentum equation is

$$Mv_{before} - mv_{before} = MV_{after} + mv_{after}$$

The conservation of kinetic energy equation is

$$\frac{1}{2}mv_{before}^2 + \frac{1}{2}Mv_{before}^2 = \frac{1}{2}mv_{after}^2 + \frac{1}{2}MV_{after}^2$$

We know v_{before}, but we don't know v_{after} or V_{after}. We have two equations which relate them, so we have two equations, two unknowns. The algebra to solve them takes a little while, so I'll skip it. The result is that

$$v_{after} = v_{before}\frac{3M - m}{M + m}$$

$$V_{after} = v_{before}\frac{M-3m}{M+m}$$

Thus if M is much larger than m, $v_{after} = 3v_{before}$. We can now see how high the little ball goes by using conservation of energy again. It bounces up at v_{after}, so its final height is given by

$$mgh_{final} = \frac{1}{2}mv_{after}^2 = \frac{1}{2}m(3v_{before})^2$$

Thus

$$h_{final} = \frac{9}{2g}v_{before}^2 = \frac{9}{2g}(2gh) = 9h$$

So in the absence of friction and if the big ball is much heavier, it bounces 9 times as high!