Lecture 13

- Center of mass
- Center of gravity

Cutnell+Johnson: 7.5, 9.3

Center of Mass

As I emphasized in the last lecture, conservation of momentum follows from Newton's first and third laws. When there are no external forces, the objects remain "as is" in the sense that the momentum is conserved. However, "as is" is a bit funny here, since we saw in the first problem above that zero momentum overall does not mean that nothing is moving. To make it clearer what's going on, we introduce the *center of mass*. Roughly speaking, the center of mass is the point at which you can balance all the objects. If we hang an object from one point, it will hang so that the center of mass is directly below the hanging point. This then lets you draw an appropriate force diagram: tension up, and gravity down. Thus the *center of gravity* is the same as the center of mass. This trick lets you easily find the center of mass, by hanging the same object two or more different ways.

Say you've got a bunch of objects of masses m_1, $m_2, \ldots$ moving around with velocities of $\vec{v}_1$, $\vec{v}_2, \ldots$. Then the total momentum of these objects is

$$\vec{p}_{total} = m_1\vec{v_1} + m_2\vec{v_2} + \ldots$$

The object as a whole has a mass $m_{total} = (m_1 + m_2 + \ldots)$. Then you can think of the object as a *whole* as moving at the center-of-mass velocity

$$\vec{v}_{CM} = \frac{\vec{p}_{total}}{m_{total}}$$

Just think of what happens when you run. Your hair may be flopping up and down, your lungs are expanding and contracting, your blood is moving up and down and left and right, and so on. Nevertheless, we still speak of a velocity of the body as a whole. This is the center of mass velocity.

To get the actual location of the center of mass, think about two blocks attached by a compressed spring. The initial center of mass is obviously in between the two blocks. There are no external forces, and the initial momentum is zero. Thus in this problem $v_{CM} = 0$ always: the center of mass never moves. It always stays in the center, even when the blocks move outward. The lighter block moves faster than the heavier one, so they still balance in the center.

The formula for the center of mass of a bunch of objects is easy to write down. The definition of velocity means that

$$\vec{p}_{total} = m_1 \frac{\Delta \vec{r_1}}{\Delta t} + m_2 \frac{\Delta \vec{r_2}}{\Delta t} + \dots$$

Also using the definition of velocity, we can write the velocity of the center of mass is

$$\vec{v}_{CM} = \frac{\Delta \vec{r}_{CM}}{\Delta t}$$

Now look at the equation we used above: $\vec{v}_{CM} = \vec{p}_{tot}/m_{tot}$. Notice that both sides are divided by Δt. Thus we can cancel this out, and we get

$$\Delta \vec{r}_{CM} = \frac{m_1 \Delta \vec{r}_1 + m_2 \Delta \vec{r}_2 + \dots}{m_{tot}}$$

This is true for any time interval (as long as the particles are moving at constant velocity). Thus we must have

$$\vec{r}_{CM} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \dots}{m_{tot}}$$

For example, the center of mass of two particles both lying on the x axis is at

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Problem In the spring problem in the last lecture, we had two blocks attached with a spring, which uncompresses. Prove that the center of mass doesn't move, with say $m_1 = 3m_2$.

Answer Say that block 1 is initially at location $x_1 = 0$, while the second block is some distance away at $x_2 = a$. Then the center of mass is at

$$x_{CM} = \frac{3m_2 \times 0 + m_2 a}{3m_2 + m_2} = \frac{a}{4}$$

Notice how the center of mass is closer to the heavier block. We then let the spring uncompress at time $t = 0$. We showed in the last lecture that because of conservation of momentum, after the spring uncompresses, $3v_1 = -v_2$ (the lighter block moves faster). There is no acceleration after the initial spring outward, so these velocities remain constant. The definition of velocity is $v_x = \Delta x/\Delta t$. Thus here

$$x_1 = v_1 t = -v_2 t/3$$

and

$$x_2 = a + v_2 t$$

Notice how you need to include the constant piece to get the right answer $x_2 = a$ at time $t = 0$. Now we can use the formula for the center of mass at times later than $t = 0$. Then we have

$$x + CM = \frac{m_1 x_1 + m_2 x_2}{m_{tot}} = \frac{m_1 v_1 t + m_2 v_2 t}{m_1 + m_2} = \frac{(-3m_2)(v_2 t/3) + m_2 v_2 t)}{4m_2} = \frac{a}{4}$$

Therefore x_{COM} always stays at $a/4$, as it must, because the momentum of the center of mass is zero.

Problem Let's go back to the car crash problem from a few lectures ago ($1000\,kg$ moving north at $13.4m/s$, $2000\,kg$ moving east at $22.4m/s$, they stick together after the collision). How does the center of mass move?

Answer In this problem, there are no external forces, so the momentum is always what we worked out last time, $\vec{p} = (4.48 \times 10^4\,kg\,m/s, 1.34 \times 10^4\,kg\,m/s)$. This is true before and after the collision. Thus the center of mass always has velocity

$$\vec{v}_{COM} = \frac{\vec{p}}{m_{total}} = (14.9m/s, 4.47m/s)$$

This is obvious after the collision, since both cars are stuck together. However, this is the velocity of the center of mass before the collision as well: because there are no external forces, the overall momentum and hence the velocity of the center of mass remains constant. The location of the center of mass is obvious after the collision as well, since the cars are stuck together. Letting the collision occur at $\vec{r} = (0, 0)$ at time $t = 0$, the position of the center of mass measure in meters is

$$\vec{r}_{COM} = (14.9t, 4.47t)$$

where t is measured in seconds. This formula is obviously true after the collision. But since the velocity of the center of mass does not change in time (there are no external forces), it is true before the collision (where $t < 0$). Thus the center of mass here always moves in a straight line.

Center of Gravity

As I said at the beginning of this lecture, the center of gravity is the same as the center of mass. The key point is that the force of gravity on the object as a whole acts at the center of gravity. This is why it's possible to do some of these tricks like hanging forks off of a wine glass: as long as the center of gravity is over a place which can support it, the object won't fall.