Lecture 14

- More on rotation
- Rotational Kinematics
- Rolling Motion
- Torque

Cutnell+Johnson: 8.1-8.6, 9.1

More on rotation

We've done a little on rotation, discussing objects moving in a circle at constant speed, and learning about centripetal force. Not surprisingly, life can be more complicated. Objects moving in a circle don't have to move at constant speed. They can have what is called *tangential acceleration*. This means that the angular speed (measured for example in revolutions per second) is changing. For example, a CD player tangentially decelerates as the laser moves to the outside, in order to preserve the same speed in m/s.

To study these sorts of problems, it is very useful to introduce a new way of describing angular distance and speed. Degrees are fairly inconvenient: the fact that the circle is divided into 360 degrees is completely arbitrary and not very intuitive. What is natural is the fact that the circumference of a circle is 2π times the radius. This suggests that instead of dividing a circle into 360 parts, we divide it into 2π parts. These parts are called *radians*. Thus 90^o corresponds to $\pi/2$ radians, 45^o corresponds to $\pi/4$ radians, 60^o corresponds to $\pi/3$ radians, etc. Radians make a number of things easier. If an object travels around a circle 3.5 times, it is much easier to say that it has traveled 7π radians than 1240 degrees. Another thing is that for small angles, you can approximate $\sin\theta \approx \theta$ if (and only if) the angle θ is measured in radians. But where it makes life really easy is when we study velocities and forces. In fact, we'll find formulas which only make sense if the angular speed is measured in radians.

The angular speed is denoted by ω. It is defined just like ordinary speed: an object which moves from an angle θ_i to an angle θ_f has average angular speed

$$\overline{\omega} = \frac{\Delta\theta}{\Delta t}$$

where as always $\Delta\theta = \theta_f - \theta_i$. Instantaneous angular speed is obtained by taking Δt small. An object moving 45 rpm has $\omega = 90\pi$ radians/minute. As we saw before, an object moving in a circle at constant speed travels a distance $2\pi r$ per revolution. Moreover, one revolution corresponds to 2π radians. Thus the speed v is related to the angular speed ω by the simple formula

$$v = \omega r$$

This formula is only valid if ω is expressed in radians. In fact, this is the main reason we define radians: it lets us write equations like 2π in equations like this. Notice that radians really aren't a unit, in the sense that v is still measured in m/s or mph or whatever: we don't need to put a radian. Note that now we can write centripetal acceleration as

$$a_c = \omega^2 r$$

Recall that the direction of centripetal acceleration and force is inward (radial). If there's an acceleration tangential to the circle, then the angular speed must change. Angular acceleration α is defined as

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

For constant angular speed, $\Delta\omega = 0$ and hence $\alpha = 0$. For constant angular acceleration, notice that objects further out have a greater tangential acceleration. In other words,

$$a_T = \alpha r$$

(again α must be measured in radians). The direction of this acceleration a_T is tangential to the circle and so is always perpendicular to the centripetal acceleration. Notice all points on a rigid object have the same angular acceleration, but that as you go further out, the acceleration a gets larger.

Rotational Kinematics

We can now easily do the kinematics of angular motion, just like we did the kinematics of linear motion in chapters 2. (Kinematics means that we don't worry about the forces, we just study the motion itself. Linear means motion on a line, as opposed to motion on a circle). From the definition of angular speed, we see that for constant angular acceleration,

$$\omega = \omega_0 + \alpha t$$

just like the earlier $v = v_0 + at$. In fact, all the linear formulas have angular versions. For example, to find the angular position of a particle of a particle undergoing a constant angular acceleration, you can use

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

The other equations also apply if you replace x with θ, v with ω, and a with α.

Problem The angular speed of a helicopter blade increases from 1 rad/s to 64 rad/s in 3 seconds with constant angular acceleration. What angle has the blade turned through in this time? What is the angular acceleration?

Answer The average angular acceleration is given by

$$\alpha = \frac{\Delta\omega}{\Delta t} = \frac{64 \text{ rad}/s - 1 \text{ rad}/s}{3\,s} = 21 \text{ rad}/s^2$$

To get the angular displacement, use

$$\begin{aligned} \theta &= \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= (1 \text{ rad}/s)(3\,s) + \frac{1}{2}(21 \text{ rad}/s^2)(3\,s)^2 = 98 \text{ rad} \end{aligned}$$

Because 2π rad =1 revolution, 98 radians is about 15 revolutions.

Rolling motion

Rolling motion is very familiar, but a little bit subtle. A bicycle obviously is moving forward, but the wheel is going around and around. To confuse matters further, it seems as if friction should be pushing the bike backwards, not forwards. Yet obviously, if there were no friction, the bike would go nowhere (if you've ever hit an ice patch on your bike, you're aware of this). So how are the forces working?

Let's look at a bike moving forward at some (linear) speed v. Since the axle is attached to the bike, the axle is obviously moving forward at a velocity v. But what is the wheel doing? The key thing to realize that (if the wheel doesn't slip), when the wheel of radius r rotates once, the car moves forward a distance $2\pi r$. This is obvious if you grab the end of a roll of tape and unroll it. Thus the linear speed of the car v is given by

$$v = \omega r$$

where ω is the angular speed of the wheel from the point of view of the car. This is our usual relation, but here it means a little more: we've related the speed of the car how fast the wheel is rotating. This means that the linear speed of the top of the wheel is $2v$ with respect to the ground: v from the fact that the axle is moving forward at a speed v, and v from the rotation. This means, more surprisingly, that the part of the wheel touching the ground is *not moving* with respect to the ground. But when you think of it, this is why friction is a forward force. The bottom of the wheel is trying to slide backward, and friction is stopping it.

One consequence is that it's the coefficient of *static* friction which is relevant to the rolling motion if the wheel is not slipping. Even though the car is moving, the part of the wheel touching the ground is not (instantaneously). This is why if you're in a car which slips, it's hard to gain traction again without slowing down substantially. Once you've slipped, it's the coefficient of kinetic friction which applies, and that's lower than the coefficient of static friction. Hence the frictional force is lower once you start slipping.

Problem On a bicycle, the gears next to the petal are a radius r_1, and on the back wheel, they are a radius r_2. The wheel is a radius r_w. Relate the linear speed v of the bicycle to the angular speed at which you pedal.

Answer A bike is built so that the front gears go at the same angular speed as the petals, while the back gears go at the same angular speed as the wheel when you are pedaling. (If you stop pedaling, the gears disengage from the wheel.) The chain connects the front gears to the back gears, so the chain must have the same *linear* velocity at both the front and the back gears. Call this linear velocity v_{chain}. Then we have

$$v_{chain} = \omega_{petal} r_1$$

for the front gear, and

$$v_{chain} = \omega_{wheel} r_2$$

Combining the two means that $\omega_{petal} r_1 = \omega_{wheel} r_2$. Now we need to get the speed of the bike. As we saw from rolling motion, the linear speed of the bike is related to the angular speed of the wheels by $v = \omega r$. As I said, on a bike, the angular speed of the wheel is the same as the angular speed of the rear gear. Thus

$$v = \omega_{wheel} r_w$$

However, we've shown that $\omega_{wheel} = \omega_{petal} r_1 / r_2$. Thus

$$v = \omega_{petal} \frac{r_w r_1}{r_2}$$

Changing the gears on the bike changes the ratio r_1/r_2. Changing to a smaller gear in the front, or a larger one in back, (i.e. decreasing r_1/r_2) makes it easier to petal. The reason is for a fixed angular speed ω_{petal}, it makes v smaller. The lower v, the less work you're doing.

Rotational Dynamics

We've seen that describing circular motion and linear motion is very similar. For linear motion, we have position x, speed v, and acceleration v. For circular motion, we have angle θ, angular speed ω, and angular acceleration α. For the next few lectures, we'll discuss forces on rotating bodies (the technical word is *dynamics*, as opposed to the study of motion, which is called *kinematics*. The new quantities we'll introduce are similar to linear quantities as follows:

Torque $\leftrightarrow$ Force
Moment of Inertia $\leftrightarrow$ Mass
Angular Momentum $\leftrightarrow$ Momentum

There are some things about rotational dynamics which get a little bit tricky (in particular, computing the moment of inertia), but on the whole, it is very similar to what we've already done with linear motion.

Torque

You've probably noticed that when you open a door, it's hard to open if you push close to the hinges. The farther you are from the hinges, the easier it is. What "easier" and "harder" mean precisely here is that it requires more force to give the door the same angular speed if you're pushing near the hinges than if you're at the other end. In other word, you get a better angular acceleration by pushing further out. You've also probably noticed that if you're using a wrench to loosen a bolt, the longer the wrench is, the easier it is to loosen the bolt. In other words, it takes less force to loosen the bolt if you're pushing the wrench farther away from the bolt.

So obviously to understand rotation fully we need to introduce something beyond force, to take into account the effect of different radii. We first need an *center of rotation*. This is really something we've already used, but we just haven't made this definition. The center of rotation is the point around which the rotation happens. The thing we've been calling the radius r in this language is just the distance from the center of rotation. Your book discusses the *axis of rotation*. I'm using a different word because we've generally reduced the rotation to a two-dimensional problem (and we'll continue to do so). Obviously a door is three dimensional. However, if you look at the door's rotation from above (an aerial view), you see that its rotation is basically a two-dimensional problem. The axis of rotation is just the extension of the center of rotation to the third dimension. For example, the axis of rotation of the door in the door is the line going through the hinges. From the aerial view, this line just looks like a point, the center of rotation.

The idea behind a torque is that applying forces can cause rotation. In other words, just like applying a force causes linear acceleration, applying a torque causes an angular acceleration. As I tried to illustrate with the examples of the door and the wrench earlier, the torque should somehow be related to the radius, just like we've already seen that $\omega = v/r$ and $\alpha = a_T/r$, where r is the distance from the center of rotation. To define torque, consider applying a force a distance r from the center of rotation. The magnitude of torque is then

$$\tau = F_T r$$

F_T is the component of the force perpendicular to the radius. The reason we need to only include the tangential component of the force is fairly obvious. Torque is a vector, but its direction gets confusing (it's along the axis of rotation, more or less), so we're not going to worry about that. What you do need to be aware of is that torque has a sign. Just like for linear motion the sign of the velocity meant the direction of the motion, for rotation, the sign of the torque indicates the direction of rotation. By convention, we choose a counter-clockwise rotation to be positive torque, and clockwise rotation to be negative torque.

So notice that the torque depends on the tangential force. Thus the torque is completely unrelated to the centripetal force. The centripetal force is what keeps the object moving in a circle. The torque is related to whether the angular speed is increasing or decreasing.

Another thing to notice about torque is that the larger the radius, the larger the torque. Let's go back to the example of the door. Say you apply a force $50\,N$ right at the hinge. The hinge is the center of rotation, so the torque in this case is zero. Say you now apply the same force close to the hinge, only a centimeter away. Then the torque is

$$\tau = (50\,N)(.01\,m) = .5N\,m$$

On the other hand, if you apply the same force at the end of the door (say $.8\,m$ away), the torque is

$$\tau = (50\,N)(.8\,m) = 40N\,m$$

Thus even though the forces are the same in these cases, the torques are very different. This is the start of the explanation why it's easier to push a door at its end; we'll come back to this issue.

By the way, this is the first time I've thought your book gave a really lousy explanation of something. Instead of defining torque the way I just did, they define these things called the lever arm and the line of action to get the torque. The book's explanation is not wrong, but I think it's not the easiest way to understand torque. This doesn't mean that you shouldn't read the book, in fact hearing something two different ways is often useful. (But if you find the book's explanation clearer than mine, please let me know!).