Lecture 15

- Torque
- Center of Gravity
- Rotational Equilibrium

Cutnell+Johnson: 9.1 - 9.3

Last time we saw that describing circular motion and linear motion is very similar. For linear motion, we have position x, speed v, and acceleration v. For circular motion, we have angle θ, angular speed ω, and angular acceleration α. For the next few lectures, we'll discuss forces on rotating bodies (the technical word is *dynamics*, as opposed to the study of motion, which is called *kinematics*). The new quantities we'll introduce are similar to linear quantities as follows:

Torque $\leftrightarrow$ Force
Moment of Inertia $\leftrightarrow$ Mass
Angular Momentum $\leftrightarrow$ Momentum

There are some things about rotational dynamics which get a little bit tricky (in particular, computing the moment of inertia), but on the whole, it is very similar to what we've already done with linear motion.

Torque

You've probably noticed that when you open a door, it's hard to open if you push close to the hinges. The farther you are from the hinges, the easier it is. What "easier" and "harder" mean precisely here is that it requires more force to give the door the same angular speed if you're pushing near the hinges than if you're at the other end. In other word, you get a better angular acceleration by pushing further out. You've also probably noticed that if you're using a wrench to loosen a bolt, the longer the wrench is, the easier it is to loosen the bolt. In other words, it takes less force to loosen the bolt if you're pushing the wrench farther away from the bolt.

So obviously to understand rotation fully we need to introduce something beyond force, to take into account the effect of different radii. We first need an *center of rotation*. This is really something we've already used, but we just haven't made this definition. The center of rotation is the point around which the rotation happens. The thing we've been calling the radius r in this language is just the distance from the center of rotation. Your book discusses the *axis of rotation*. I'm using a different word because we've generally reduced the rotation to a two-dimensional problem (and we'll continue to do so). Obviously a door is three dimensional. However, if you look at the door's rotation from above (an aerial view), you see that its rotation is basically a two-dimensional problem. The axis of rotation is just the extension of the center of rotation to the third dimension. For example, the axis of rotation of the door in the door is the line going through the hinges. From the aerial view, this line just looks like a point, the center of rotation.

The idea behind a torque is that applying forces can cause rotation. In other words, just like applying a force causes linear acceleration, applying a torque causes an angular acceleration. As I tried to illustrate with the examples of the door and the wrench earlier, the torque should somehow be related to the radius, just like we've already seen that $\omega = v/r$ and $\alpha = a_T/r$, where r is the distance from the center of rotation. To define torque, consider applying a force a distance r from the center of rotation. The magnitude of torque is then

$$\tau = F_T r$$

F_T is the component of the force perpendicular to the radius. The reason we need to only include the tangential component of the force is fairly obvious. Torque is a vector, but its direction gets confusing (it's along the axis of rotation, more or less), so we're not going to worry about that. What you do need to be aware of is that torque has a sign. Just like for linear motion the sign of the velocity meant the direction of the motion, for rotation, the sign of the torque indicates the direction of rotation. By convention, we choose a counter-clockwise rotation to be positive torque, and clockwise rotation to be negative torque.

So notice that the torque depends on the tangential force. Thus the torque is completely unrelated to the centripetal force. The centripetal force is what keeps the object moving in a circle. The torque is related to whether the angular speed is increasing or decreasing.

Another thing to notice about torque is that the larger the radius, the larger the torque. Let's go back to the example of the door. Say you apply a force $50\,N$ right at the hinge. The hinge is the center of rotation, so the torque in this case is zero. Say you now apply the same force close to the hinge, only a centimeter away. Then the torque is

$$\tau = (50\,N)(.01\,m) = .5N\,m$$

On the other hand, if you apply the same force at the end of the door (say $.8\,m$ away), the torque is

$$\tau = (50\,N)(.8\,m) = 40N\,m$$

Thus even though the forces are the same in these cases, the torques are very different. This is the start of the explanation why it's easier to push a door at its end; we'll come back to this issue.

By the way, this is the first time I've thought your book gave a really lousy explanation of something. Instead of defining torque the way I just did, they define these things called the lever arm and the line of action to get the torque. The book's explanation is not wrong, but I think it's not the easiest way to understand torque. This doesn't mean that you shouldn't read the book, in fact hearing something two different ways is often useful. (But if you find the book's explanation clearer than mine, please let me know!).

Center of Gravity

We saw a few lectures ago how to define the center of mass. I also pointed out that the center of gravity was the same as the center of mass. Now we can define the center of gravity a different (but equivalent) way. The center of gravity is the point in an extended object at which all the torques due to gravity vanish. This is why you can balance an object at the center of gravity. Let's quickly rederive the equation for the center of mass/gravity for two objects. Say we have a barbell, with two large weights m_1 and m_2 at opposite ends. Say these weights are so heavy that we can neglect the mass of the bar itself. The first weight is at a position x_1, and the second at a position x_2. Let's compute the torques around some arbitrary point on the bar, call it X. Then

$$\sum \tau = m_1 g(X - x_1) - m_2 g(x_2 - X)$$

If this point X is the center of mass/gravity, we have

$$\sum \tau = 0 \text{ if } X = x_{CM}$$

Rearranging the terms says that

$$x_{CM} m_1 + x_{CM} m_2 - m_1 x_1 - m_2 x_2 = 0$$

Solving for x_{CM} gives our earlier formula

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

The reason we bother to define the center of gravity as well as the center of mass is that it makes one crucial fact clear. When computing torques due to an extended body, you can treat the weight of the body as acting at the center of gravity. We'll use this fact in the example of a ladder leaning against a wall. Even though the weight of the ladder is spread out over the whole ladder, for the purposes of computing the torque due to the ladder's weight, you can think of it as acting entirely from the center.

Rotational Equilibrium

We'll see soon precisely how applying a torque causes an angular acceleration. But let's first discuss a case important to architects, equilibrium. So let's go back and review what equilibrium means. Before we defined equilibrium as a situation where the net forces vanish:

$$\sum \vec{F} = 0$$

This doesn't necessarily mean static: static means that not only do the forces vanish but the velocities vanish as well.

Now we can introduce an additional form of equilibrium. *Rotational equilibrium* means that the net external torque vanishes:

$$\sum \tau = 0$$

Generally, people (including your book) say that an object is in equilibrium if both the net external force $\sum \vec{F} = 0$ and the net external torque $\sum \vec{\tau} = 0$. It's easy to see that just because the net force vanishes does not necessarily mean that that the net torque vanishes. Think about two people pushing on a revolving door with the same forces. If they're on opposite sides, they're pushing in opposite direction, so the net force is zero. However, the torques they're applying are both the same (either clockwise or counterclockwise). Therefore, these torques add together, not cancel like the net force.

One thing to note about rotational equilibrium is that since there are no external torques, you can put the axis of rotation anywhere. The torque must vanish, no matter where you put it. For doing problems, it is usually convenient to put the center of rotation so that one of the forces goes through it. Thus this particular torque will vanish, making the computation a little easier.

Now we can apply the equations $\sum \vec{F} = 0$ and $\sum \vec{\tau} = 0$ to a variety of situations.

Problem A $4.0\,m$ long diving board is supported by a fulcrum and a bolt (see Figure 9.6 in the book). The bolt is at one end, while the fulcrum is $1.5m$ from the bolt. Say a force of $4000\,N$ will dislodge the bolt. What is the maximum possible weight of a diver, for the bolt not to pull out? Neglect the weight and the bending of the board itself.

Answer The forces on the board are easy to draw. On one end, the weight W of the diver is downward. On the other end is the force of the bolt F_b, also downward. The fulcrum is pushing upward with a force F_f. The sum of the forces must be zero. Thus

$$\sum F = W + F_b - F_f = 0$$

Let's use the fulcrum as the center of rotation. Then the radius of bolt is $r_b = 1.5\,m$, while the radius of the diver is $r_d = 2.5\,m$. The radius of the fulcrum itself is zero, so it causes no torque around this axis. The sum of the torques

$$\sum \tau = F_b r_b - W r_d = 0$$

Notice that the sign is different for these two terms. This is because the torque due to the diver is clockwise, while the torque due to the bolt is counterclockwise. Now to solve our problem. The two equations we have just derived allow us to relate the force on the bolt F_b to the weight of the diver. In fact, all we need for this particular problem is the torque equation: it says that

$$W = F_b \frac{r_b}{r_d} = .6 F_b$$

Thus the maximal force on the bolt of $4000\,N$ is caused by a weight of $2400 N$ (which corresponds to about 500 pounds).

Problem A ladder of length $8.0\,m$ and weight $350\,N$ is leaning against a smooth wall at an angle of 60^o ($\pi/3$ rads) with the horizontal. (Smooth means the wall does not exert any frictional force, i.e. $\mu_k = \mu_s = 0$). A person of mass $90\,kg$ stands 3/4 of the way up the ladder. What frictional force does the ground need to apply to prevent the ladder from sliding?

Answer Let's start by drawing the force diagram. Because the wall is smooth, it has no force in the vertical direction, but it does still exert a normal force F_w outward. The ground is not frictionless (if it were the ladder would instantly fall), so it exerts a normal force F_g upward, and a frictional force F_f to the right. There are two weights W_p (the person) and W_l (the ladder) in the problem, both of course pointing downward. Since the ladder is in equilibrium, the net force and the net torque must vanish. There are two force equations, for the x and y directions:

$$\sum F_x = F_f - F_w = 0$$

$$\sum F_y = F_g - W_l - W_p = 0$$

Notice that there are no sines or cosines here: all the forces are in the x or y direction. There is one more equation, because the net torque must vanish as well. This is true no matter which point we choose to be the center of rotation. Let's pick the point on the ground. Then there is no torque associated with friction F_f and the F_g, because these forces are at the center. It is important to remember that the force which goes into the torque equation is the tangential force. The weights are both 60^o off the tangent, while the normal force is 30^o off the tangent. Also, the radius enters in the torque equation. You can think of the weight of the ladder as acting at the center of mass, so this distance is $4.0m$ from the center of rotation. The person is $6.0\,m$, while the wall is $8.0\,m$. Thus the torque equation is

$$\sum \tau = (8.0m) F_w \cos(30^o) - (6.0\,m) W_p \cos(60^o) - (4.0\,m) W_l \cos(60^o) = 0$$

Note the signs: the force of the wall is counter-clockwise, while the weights are clockwise. Since we know the weights of the ladder and the person, this says that the force of the wall is

$$F_w = \frac{(6.0\,m)(90\,kg)(9.8\,m/s^2)\frac{1}{2} + (4.0\,m)(350\,N)\frac{1}{2}}{(8.0\,m)\sqrt{3}/2} = 480\,N$$

Moreover, the x component of the force equation gives

$$F_f = F_w = 480\,N$$

Thus we didn't need the y force equation: it merely says that the normal force up is equal to the weights going down.