Lecture 16

- Newton's Second Law for Rotation
- Moment of Inertia
- Angular momentum

Cutnell+Johnson: 9.4, 9.6

Newton's Second Law for Rotation

Newton's second law says how a net force causes an acceleration. We can apply this to a rotation. Let's look at a toy airplane tied to the end of a string. We talked at length how the centripetal force is necessary to keep the plane moving in a circle. Centripetal force is all you need to keep the plane moving at a constant angular speed. However, if you want to change the angular speed (to get it going faster or slower), you need to apply a tangential force as well. In other words, you need to apply a torque.

What we'll do here is see how the torque is related to the angular acceleration. In other words, we'll derive the rotational version of Newton's second law. Let's first consider a case where all the mass is in one place, like the airplane of mass M on a string of length r traveling in a circle. The only tangential force is the force due to the airplane engine: the string only provides a centripetal force. Call this force F_T. Newton's second law (which always applies in any situation) says that

$$F_T = ma_T$$

We can now compute the torque:

$$\tau = F_T r = ma_T r$$

We derived a few lectures ago that for a single object, the angular acceleration is related to the linear acceleration by

$$a_T = \alpha r$$

Plugging this into the equation for the torque:

$$\tau = (mr^2)\alpha$$

The factor mr^2 is called the *moment of inertia*. The equation I wrote applies just for a single object. The general equation is

$$\tau = I\alpha$$

For a single object, the moment of inertia I is equal to mr^2.

I'll show below that the angular momentum L is given by

$$L = I\omega$$

Just like a torque is related to a force, but is not the same thing, angular momentum is related to the usual momentum but is not the same thing. Thus you do not add it with the linear momentum: in fact, L has dimensions (mass)(distance)^2/(time), while linear momentum p has dimensions (mass)(distance)/(time). If there is no net external force, the linear momentum is conserved. If there is no net external torque, angular momentum is conserved.

Moment of inertia for an extended object

Let's now consider two airplanes tied to various points on the same string. Then just repeat the above argument to get the torque due to the first airplane:

$$\tau_1 = m_1(r_1)^2\alpha$$

where r_1 is the distance between the first plane and the center of rotation. The same sort of relation holds for the second airplane:

$$\tau_2 = m_2(r_2)^2\alpha$$

Note that the angular acceleration for the two is the same because they are tied to the same string. Thus the total torque is

$$\sum \tau = \tau_1 + \tau_2 = (m_1(r_1)^2 + m_2(r_2)^2)\alpha$$

Thus the combined moments of inertia for the two planes is $I = m_1(r_1)^2 + m_2(r_2)^2$. Doing this for any collection of objects, we have

$$I = \sum m_i(r_i)^2$$

It is important to note that these objects *must* be bound together and so are moving at the same angular acceleration α. If this is true, we can speak of the combined moment of inertia of all the objects, and then the relation

$$\sum \tau = I\alpha$$

is true. This is the general relation between torque and angular acceleration. Thus we see that the moment of inertia plays the role of the mass here. One important thing to note: just like the torque depends on where the center of rotation is, the moment of inertia also depends on the location of the center of rotation.

So far, I hope this is nothing too complicated. Where understanding the moment of inertia gets a little trickier is when the object is extended and rigid. An example of a rigid object is a door swinging. The distance from the center of rotation depends on where you are on the door. Computing the center of inertia requires adding up the contributions from all of the door. The farther out the mass is, the greater the contribution to the moment of inertia. So let's look at a door of mass M and width R. If all the mass of the door were concentrated at the far end, the momenta of inertia would be MR^2. Say half the mass were concentrated at the midpoint, and the other half at the end. Then the moment of inertia would be

$$\frac{M}{2}\left(\frac{R}{2}\right)^2 + \frac{M}{2}R^2 = \frac{5}{8}MR^2$$

For a normal door, the mass is evenly distributed. Its moment of inertia is

$$I_{door} = \frac{1}{3}MR^2$$

To really compute this you need to do an integral. You don't need to know how to do this for this class, but for those of you who like calculus, it's a simple integral:

$$\frac{M}{R}\int_0^R r^2 dr = \frac{1}{3}MR^2$$

There's a table in section 9.4 of your book of the moments of inertia for various types of objects. Some others are $I = (2/5)MR^2$ for a solid ball rotating around an axis through the center, $I = MR^2/2$ for a solid disk, and so on.

Problem Does a hollow object or a solid object roll faster down a hill?

Answer If the two objects are the same size and mass, then the external torques will be the same. The hollow object has a larger moment of inertia. Thus for a given torque, its angular acceleration will be *less*. This remains true even if the objects have different masses. This is because the mass cancels out of the equation: the torque is proportional to the mass, as is the moment of inertia. In fact, we will see next time that the speed at which something rolls down the hill is independent of the radius as well. However, it is *not* independent of the shape.

Problem. Say we attach a board of length $.50\,m$ to a hinge on the ground. We raise it 30^o above the horizontal and drop it. What is its angular acceleration right after it is dropped? What is it just before it hits the ground?

Answer The key is to remember that gravity acts at the center of mass. So there is a force of mg downward on the board at its center point. However, we must divide this force into radial and tangential components. The radial component contributes to the centripetal forces (as does the hinge). The tangential component is what causes the angular acceleration. This force is

$Mg\cos(30^o) = Mg\cos(\pi/6) = Mg\sqrt{3}/2$, and so the torque is

$$\tau = Mg\frac{\sqrt{3}}{2}(.25\,m)$$

The moment of inertia of the board is just like that of a door, so $I = MR^2/3$. Thus the angular acceleration here is

$$\alpha = \frac{\tau}{I} = \frac{g(\sqrt{3}/2)(.25\,m)}{(.50\,m)^2/3} = \frac{(9.8m/s^2)(.866)}{1.0m/3} = 25rad/s^2$$

Right before it hits the ground, the board is horizontal. Now the force of gravity is perpendicular to the board, and so is tangential. The solution is the same as above if you omit the $\cos(30^o) = .866$, so the angular acceleration is $\alpha = 29$ rad$/s^2$.

Problem What would the moment of inertia be for a door of width R if the center of rotation were around the middle (i.e. a revolving door)?

Answer Just think of a revolving door as two doors, of width $R/2$ and mass $M/2$ each, attached in the center. Then you can use the formula for a door at the end, and add the two together. The moment of inertia is

$$I = \frac{1}{3}\frac{M}{2}\left(\frac{R}{2}\right)^2 + \frac{1}{3}\frac{M}{2}\left(\frac{R}{2}\right)^2 = \frac{1}{12}MR^2$$

You can check that this indeed is the formula in your book. I did this example just to illustrate that there shouldn't be anything mysterious about those formulas in your book. They all come from arguments like this, with maybe a little calculus.

Angular Momentum

The angular momentum of a rotating object is

$$L = I\omega$$

Let's see where this comes from. Remember that torque is $\tau = I\alpha$. Let's relate the torque to the angular momentum, using the fact that the definition of angular acceleration is

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

This means that

$$\tau = I\alpha = I\frac{\Delta\omega}{\Delta t} = \frac{\Delta L}{\Delta t}$$

In the last step I used the fact that the inertia isn't changing, so $\Delta L = \Delta(I\omega) = I\Delta\omega$. So this tell us that a torque causes the angular momentum to change. This is just like the linear case. There we have

$$F = ma = m\frac{\Delta v}{\Delta t} = \frac{\Delta p}{\Delta t}$$

A force changes the momentum, a torque changes the angular momentum.

From these formulas it follows that if there are no external forces, $\Delta p = 0$: the momentum does not change. Likewise, if there are no external torques, the angular momentum does not change. The demonstrations on the spinning stool relied on this fact. There are no external torques on the stool (neglecting the friction in the stool itself). Thus the total angular momentum is conserved. We already saw that if you decrease the moment of inertia by pulling weights in, the angular speed must increase. If I goes down, ω must go up to keep $L = I\omega$ constant.

Problem Two identical spaceships are attached by a massless cable. The spaceships are rotating at a speed $v_0 = 17m/s$ around their center of mass. They both pull in the cable until the distance between them is reduced by a factor of two. What is their final speed?

Answer The ships (each of mass M) are identical, so the center of mass is halfway in between them. Call the distance to the center of mass R, so that the initial angular velocity $\omega_0 = v_0/R$. Then moment of inertial of each ship is

$$I_1 = I_2 = MR^2$$

so the angular momentum of each is

$$L_1 = L_2 = I\omega_0 = (MR^2)(v_0/R) = MRv_0$$

After they pull the lines in, their angular momentum must remain the same. However, their moment of inertia has changed: it is smaller by a factor of 4. Thus for each of them

$$L_{f1} = L_{f2} = (M(R/2)^2)\omega_f$$

Since this is the same as the initial angular momentum, we have

$$\omega_f = \frac{MRv_0}{MR^2/4} = \frac{4v_0}{R}$$

Thus its final angular speed is four times its initial angular speed. The final velocity is

$$v_f = \omega_f\left(\frac{R}{2}\right) = \frac{4v_0}{R}\left(\frac{R}{2}\right) = 2v_0 = 34\,m/s$$

This is twice the initial velocity.

One other thing about angular momentum: it is a vector. The direction of the vector is along the axis of rotation. So this is in fact how airplanes used to tell which direction they were going. The would have a wheel spinning around, on a mount which can move essentially friction free. Thus when the plane turns, the wheel keeps spinning with the same axis of rotation. This is conservation of angular momentum. Not only does the magnitude stay the same (in the absence of external torques), but the the axis of rotation must stay the same as well. This is what allowed the plane to keep track of which direction it's moving. To tell all possible directions, the plane actually needs 3 gyroscopes. Nowadays, there's this satellite system (the global positioning system) which you can even get in your cars, so the gyroscopes are obsolete.