Lecture 17

- Rotational Dynamics
- Rotational Kinetic Energy
- Stress and Strain and Springs

Cutnell+Johnson: 9.4-9.6, 10.1-10.2

Rotational Dynamics (some more)

Last time we saw that the rotational analog of Newton's 2nd Law is

$$\sum \tau = I\alpha$$

The sum of the torques plays the role of the sum of the forces, the moment of inertia plays the role of the mas, and the angular acceleration α plays the role of the acceleration. Likewise, we saw that angular momentum L is

$$L = I\omega$$

just like $p = mv$. The angular momentum is related to torque by

$$\sum \tau = \frac{\Delta L}{\Delta t}$$

just like momentum is related to force. If there are no external torques then $\Delta L = 0$: angular momentum is conserved.

All of these things are defined with respect to a center or axis of rotation. For example, in the definition of torque, the distance to the center of rotation explicitly appears. The angular speed and angular acceleration are of course defined with respect to a center of rotation. However, it is important to note that the moment of inertia is also defined with respect to a center of rotation. For example, look at the table of moments of inertia in the book. The moment of inertia of a ball with the axis through the center is $I = (2/5)MR^2$, while the moment of inertia of the same ball with the axis through the edge is $(7/5)MR^2$. If you're studying the torques around the center, you must use the first moment of inertia, while if you're studying the torques around the edge, you must use the latter.

Problem Find the linear acceleration of a ball rolling down a 30^o slope.

Answer Let's do the problem in two ways. The easiest way is in fact to look at the torques around the point at which the ball is touching the slope. The reason this is the easiest is because we don't initially know the force due to friction, and around this point, there is no torque due to the friction. The only torque around this point is due to the weight. It is

$$\tau_{edge} = F_T r = mg\sin(\theta)r$$

This torque must be equal to

$$\tau_{edge} = I_{edge}\alpha$$

where $I_{edge} = (7/5)mr^2$. The linear acceleration is related to the angular acceleration, so

$$a = \alpha r = \frac{\tau_{edge}}{I_{edge}}r = \frac{mgr\sin\theta}{(7/5)mr^2}r$$

Simplifying all this gives

$$a = \frac{5}{7}g\sin(30^o) = 3.5m/s^2$$

There are several things worth noting. The answer is independed of the mass and the radius of the object: it only depends on g, the angle, and the coefficient of the moment of inertia. If we had done this problem for a solid cylinder instead of a sphere, we would have gotten $(2/3)g\sin(\theta)$. Also, note that if the object were sliding without friction instead of rolling, the answer would have been $a = g\sin\theta$, something we derived a long time ago. Thus it slows down because of the rolling. You can think of this as happening because some of the energy goes into rotational kinetic energy instead of kinetic energy (as we will see below).

The other way to do this problem is to study the torques around the center. However, here we have to use the force equations as well, in order to determine the force due to friction. We have

$$\tau_{center} = F_{fr}r = I_{center}\alpha$$

where $I_{center} = (2/5)mr^2$. To get F_{fr}, we need to write the force equation in the direction parallel to the slope. This is exactly the same as before, namely

$$\sum F_{\parallel} = ma = mg\sin\theta - F_{fr}$$

Thus $F_{fr} = mg\sin\theta - ma$. Plugging this into the torque equation, and using $\alpha = a/r$, we get

$$I_{center}a/r = (mg\sin\theta - ma)r$$

Simplifying this gives

$$(2/5)a = g\sin\theta - a$$

so that

$$a = (5/7)g\sin\theta$$

like before. To get this to work out, it was necessary that

$$I_{edge} = I_{center} + mr^2$$

This is an example of a fancy theorem called the parallel-axis theorem, which you'll have a homework problem on.

Rotational Kinetic Energy

You may have noticed that chapter 8 was basically the rotational version of chapter 2, the chapter on linear motion. The first three sections of chapter 9 are basically the rotational version of chapter 4, the chapter on force and Newton's laws. Now we're moving up to the rotational version of chapter 6, the chapter on work and energy. As I hope you've figured out by now, all the formulas we learned before have rotational analogs.

The kinetic energy of a rotating object is also easy to get. We derived the equation $\tau = I\alpha$ before by first considering a single object moving in a circle. Then its kinetic energy is

$$KE = \frac{1}{2}mv^2$$

as always. Since the object is moving in a circle, its angular speed ω is related to its linear speed v by

$$v = \omega r$$

where r is of course the distance from the center of rotation. Using this in the equation for kinetic energy gives

$$KE = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}(mr^2)\omega^2 = \frac{1}{2}I\omega^2$$

We've just derived this for a single object, but it applies for any extended object. The reason is the same as we saw before. You can just add together the contributions of all the masses at all the locations, and get

$$KE = \frac{1}{2}I(\omega)^2$$

Of course you should remember that depending on the type of object, you need to use different formulas for I. Thus here's a simple

Problem What is the kinetic energy of the earth's rotation about its axis?

Answer The moment of inertia of a sphere of uniform density rotating around its axis is

$$I = \frac{2}{5}MR^2 = \frac{2}{5}(6.0 \times 10^{24}\,kg)(6.4 \times 10^6\,m)^2 = 9.8 \times 10^{37}\,kg \cdot m^2$$

The angular speed of the earth is 2π radians/day, which is $2\pi/(3600\times 24) = 7.3\times 10^{-5}$ radians/s. The kinetic energy of rotation therefore is

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2}(9.8 \times 10^{37}\,kg \cdot m^2)(7.3 \times 10^{-5}\ \text{rads}/s)^2 = 2.6 \times 10^{29}\,J$$

Problem A bicycle wheel has a radius of $.330\,m$ and a rim of mass $1.20\,kg$. The wheel has 50 spokes, each with a mass of $10.0\,g$. The wheel is on a bike moving at $10.0m/s$. What is its kinetic energy?

Answer First, we need to find the moments of inertia. The center of rotation is the axle. All the mass of the rim is concentrated at the radius $.33\,m$. Thus the moment of inertia of the rim is

$$I_{rim} = MR^2 = (1.20\,kg)(.330m)^2 = .131\,kg \cdot m^2$$

On the other hand, the spokes extend outward from the axle. Each spoke is like a door, so for each spoke we use the formula

$$I_{spoke} = \frac{1}{3}MR^2 = \frac{1}{3}(.010\,kg)(.330\,m)^2 = 3.6 \times 10^{-4} kg \cdot m^2$$

The total moment of inertia of the wheel is therefore

$$I = I_{rim} + 50 I_{spoke} = .149\,kg\,m^2$$

Now we need to find the energy. First, let's find the rotational energy. We need the angular speed. A few lectures ago we showed how for rolling motion, $v = \omega r$. Thus

$$\omega = \frac{10.0m/s}{.330s} = 30.3\ \text{rad}/s$$

and the *rotational* kinetic energy is

$$KE_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}(.149kg \cdot m^2)(30.3\ \text{rad}/s)^2 = 68.4J$$

This, however, is not the total kinetic energy. The reason is that the bike and hence the wheel are moving as well. The total mass of the wheel is $1.20\,kg + 50(.010\,kg) = 1.70\,kg$, so the usual kinetic energy is

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}(1.70\,kg)(10.0m/s)^2 = 85.0J$$

and the total is

$$KE = 68.4J + 85.0J = 153.4J$$

We can use this kinetic energy combined with gravitational potential energy just like we did before. In the absence of forces other than gravity, we can still use

$$\Delta KE + \Delta PE = 0$$

The only thing new this time is that we must remember to include the rotational part of the kinetic energy as well as the usual $mv^2/2$.

Problem An a ball rolls down a slope without slipping. It starts at rest at a height $h = .40m$. Assume no energy is lost to friction as it rolls. What speed does it have at the bottom?

Answer The initial potential energy is

$$PE_i = mgh$$

while the initial kinetic energy is zero. The final potential energy is zero, while the final kinetic energy is

$$KE_f = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

For an object rolling without slipping, $v = \omega r$, so the final kinetic energy is

$$KE_f = \frac{1}{2}v^2(m + \frac{I}{r^2})$$

At the bottom of the slope, the potential energy is zero. Thus the final kinetic energy is the same as the initial potential energy:

$$\begin{aligned} KE_f &= PE_i \\ \frac{1}{2}v^2(m + \frac{I}{r^2}) &= mgh \end{aligned}$$

Solving for v gives

$$v = \sqrt{\frac{2gh}{1 + I/(mr^2)}}$$

This equation is true for any object rolling down a slope, whatever I is. Note that it is independent of the mass and the radius, because $I \propto mr^2$. For a rolling ball, $I = 2mr^2/5$, so with $h = .4\,m$,

$$v = \sqrt{\frac{10gh}{7}} = 2.4\,m/s$$

Stress and Strain and Springs

So far we've dealt with rigid objects. That means if you put a net force on it, it doesn't change shape it moves. Yet of course even steel isn't completely rigid. If you squeeze it, it shrinks. If you pull on a rope, it stretches. These are examples of stress and strain. The *stress* is related to the external force you put on an object. The *strain* is resulting deformation of the object's shape or size. Stress and strain can be thought of as "cause" and "effect". You put a force on something, it changes.

This sort of deformation is called *elastic deformation*. The word “elastic” means that after you removed the external force (i.e. remove the stress), the object returns to its original shape and size. When the stress is not too large, there is a simple law relating stress and strain, which gets the name of Hooke’s Law. It is simply

Stress is proportional to strain.

In other words, if you double the stress, you double the resulting strain. Next time we will do explicit examples of this.