Lecture 18

- Stress and Strain and Springs
- Simple Harmonic Motion

Cutnell+Johnson: 10.1-10.4,10.7-10.8

Stress and Strain and Springs

So far we've dealt with rigid objects. A rigid object doesn't change shape even if you put more than one force in more than one place on it. Yet steel isn't completely rigid. If you squeeze it, it shrinks. If you pull on a rope, it stretches. These are examples of stress and strain. The *stress* is related to the external force you put on an object. The *strain* is resulting deformation of the object's shape or size. Stress and strain can be thought of as "cause" and "effect". You put a force on something, it changes.

This sort of deformation is called *elastic deformation*. The word "elastic" means that after you removed the external force (i.e. remove the stress), the object returns to its original shape and size. When the stress is not too large, there is a simple law relating stress and strain, which gets the name of Hooke's Law. It is simply

Stress is proportional to strain.

In other words, if you double the stress, you double the resulting strain.

Let's give some examples of this. Say we've got a rod of length L_0 and cross-sectional area A. We attach one end to a wall, and pull on the other end with a force F. As a result, the rod stretches by some amount ΔL. If Hooke's Law applies, we have the relation

$$\frac{F}{A} = Y\frac{\Delta L}{L_0}$$

The stress here is the force per area F/A, while the strain is the ratio of the change in length to total length $\Delta L/L_0$. As advertised, the stress is proportional to the strain. The proportionality constant here is called *Young's modulus*. Note that ΔL can be greater or less then zero. The latter is called compression, while the former is as a result of tension.

The value of Y depends on the material. There's a table in your book: for steel it's $2.0 \times 10^8 N/m^2$, for bone it actually depends slightly on whether you're pushing or pulling: it's $9.4 \times 10^9\, N/m^2$ and $1.6 \times 10^{10} N/m^2$, respectively. Note that the MKS units of stress and Young's modulus are N/m^2, while strain is dimensionless (since is a ratio).

Another type of deformation is called *shear*. An example is taking a book resting on a table and pushing across the top of it with a force F. Then the top of the book moves with the force, while the frictional force of the table holds the bottom in place. Thus shear changes the shape of the object. To define the shear modulus, we need to use Δx, the distance the top of the book moves. The strain is then given by $\Delta x/y_0$, where y_0 is the thickness of the book. Note the first is measuring a distance in the x direction, while the second is measuring a distance in the y direction. This is why shear differs from the stretching discussed above, where both ΔL and L_0 are in the same direction. For shear, the stress and strain are related by

$$\frac{F}{A} = S\frac{\Delta x}{y_0}$$

The constant of proportionality S is called the *shear modulus*. You've probably heard of shear in bolts. If you have a bolt holding to pieces of metal together, opposite forces on the pieces causes the bolt to shear. Too much shear, and the bolt breaks (the physics once the bolt breaks is no longer described by Hooke's Law).

Another property of an object which can be deformed by an external force is the volume. One example of this is a balloon. By putting compressed air or helium inside, it puts a force on the balloon and makes it large. Similarly, if you put an inflated balloon in water, the balloon shrinks in size. The reason is that the external water pressure is greater than the external air pressure. The deeper the balloon goes in the water, the greater the external pressure, and the smaller the balloon gets.

You'll notice I just started using the word pressure. The pressure P on a surface is the component of force perpendicular to the surface divided by the area over which the force acts:

$$P = \frac{F_\perp}{A}$$

For volume deformation, the stress is not exactly the pressure, like it was before. Here the stress is the change in pressure ΔP. The strain is the change in volume over the volume: $\Delta V/V_0$. Then Hooke's law applied here means that

$$\Delta P = -B\frac{\Delta V}{V_0}$$

The proportionality constant B is known as the *bulk modulus*. We included the minus sign to make B positive: an increase in external pressure decreases the volume. Again, you can find a table of bulk moduli in the book (the plural of modulus is moduli, sorry).

A very simple system which also obeys Hooke's law is the spring. Here the stress is the force you put on the spring, while the strain is the distance the string stretches. Thus Hooke's Law for springs says that the force F to stretch a string a distance x is

$$F = kx$$

where the k is known as the string constant. The MKS units of k are N/m. Note that k is always positive (it takes a positive force to stretch a string) but that x and hence F can be negative. Negative x corresponds to compressing the string. If you are interested instead in the restoring force of the spring, then $F_{restore} = -kx$. This is the force the *spring* exerts, which of course is always the opposite of the force exerted on it.

Another object which exhibits simple harmonic motion is a pendulum, as long as it isn't swinging too far. The easiest way to study the pendulum is to compute the torque. The tension causes no torque, but gravity causes a torque

$$\tau = mg \sin\theta L$$

where m is the weight of the pendulum, L the length of the string, and θ the angle the string makes with the vertical. This is true for any angle θ. We're interested in the situation where θ is small. If so, we can replace $\sin\theta$ with θ, as long as we measure θ in radians. This is obvious by drawing a picture: for small angles, the arc length is the same as the line length. So finally we have for the force in the x direction:

$$\tau = mgL\theta$$

So this is another example of Hooke's Law. The strain θ is proportional to the stress τ.

Elastic Potential Energy

All the stress-strain relations discussed above are elastic deformations. If you remove the external force, then the object tries to return to its original shape. These are examples of what we called conservative forces earlier in the course (gravity is the main example we discussed before). A conservative force is one where you have a potential energy. A potential energy means as the object moves around, it may lose kinetic energy, but it gains potential energy. So compressing a spring is like lifting something up higher. By dropping an object, you get the kinetic energy back. Similarly, by letting the spring uncompress, you get the kinetic energy back.

This kind of potential energy is called *elastic potential energy*. For a spring, the formula is

$$PE_{el} = \frac{1}{2}kx^2$$

There is a big difference between gravitational potential energy and elastic potential energy. First of all, the elastic energy is quadratic in the displacement (x^2), whereas it is linear for gravity (h). As a result, the the potential energy is always positive, whether x is positive or negative. You can get kinetic energy (which is also always positive) out of a spring by either compressing it or stretching it. Another consequence is that it gets larger much quicker.

So now we have two equations describing a spring. The first is $F = kx$, the second is $PE = \frac{1}{2}kx^2$.

Problem A spring with $k = 40N/m$ is hanging with a weight $W = mg = 12N$ attached. I lift the weight a height $.30\,m$ above the position where the string is unstretched. I drop the weight. How far downward does the spring stretch? What velocity does the weight have when the string is unstretched?

Answer You could do this problem using forces, but the easiest way is to use conservation of energy. Here there are three kinds of energy: gravitational potential energy, the elastic potential energy of the spring, and kinetic energy, so the overall energy is

$$E = mgh + \frac{1}{2}mv^2 + \frac{1}{2}kh^2$$

I have set $h = 0$ to be the position where the spring is unstretched, so h can be positive or negative. Since there are no other forces, the overall energy is conserved. The initial energy is

$$E_i = (12\,N)(.30m) + 0 + \frac{1}{2}(40\,N/m)(.30\,m)^2 = 5.4J$$

At its bottommost point the string is stretched by a distance $h = h_B$, where h_B is negative. The energy here is the same, so

$$5.4J = (12\,N)h_B + 0 + \frac{1}{2}(40\,N/m)h_B^2$$

Using the quadratic equation gives

$$h_B = -.90\,m$$

Note that there are two solutions to the quadratic equation. The one with h_B positive is the original position $h = .30\,m$ (which of course has the same energy as itself!) When the string is unstretched, the weight is moving with a speed v_u. The energy is still the same, but now is totally kinetic energy:

$$5.4\,J = 0 + \frac{1}{2}\frac{12N}{9.8m/s^2}v_u^2$$

Solving for v_u gives

$$v_u = 3.0\,m/s$$

Simple Harmonic Motion

Forces obeying Hooke's Law can be thought of as restoring forces. That means that if you disturb the system from rest, the forces act to restore the initial situation. However, what happens is that the system overshoots its initial situation. For example, if you stretch a string and let it go, it does go back to its initial position, but then it keeps on going and compresses. It then uncompresses, but again overshoots. The reason is simple. As we saw, when you stretch the string to begin with, you are putting energy into the system. The energy can be converted between kinetic and potential, but if there are no non-conservative forces like friction, the energy will not go away. The spring will just oscillate back and forth forever. This is why pendula take a long time to stop: the only non-conservative force is air friction, and that's very small.

This sort of motion is called *simple harmonic motion*. This holds for all forces obeying Hooke's Law. Let's concentrate on the case of the spring, because it is the easiest to visualize. Then by using a little calculus, one can derive precisely how this back-and-forth motion depends on time. Let's call the amount the string is stretched A. Let's define the coordinate x so that $x = 0$ at time $t = 0$, the spring is at its peak stretching A. Then the position x as a function of time is

$$x = A\cos(\omega t)$$

The constant A is called the amplitude, and is independent of time. Since the maximum value of sin is 1 and minimum is -1, this means the farthest the spring gets from its initial position $x = A$ is $x = -A$. After a time $T = 2\pi/\omega$, the object is back where it was at time zero. Let's check this in the formula. At some time t_0, the end of the spring is at at a position $x_0 = A\cos(\omega t_0) = A\cos(2\pi t_0/T)$. At a time T later, it is at a position

$$x = A\cos(\omega(t_0 + T)) = A\cos\left(\frac{2\pi}{T}(t_0 + T)\right) = A\cos\left(\frac{2\pi t_0}{T} + 2\pi\right) = A\cos\left(\frac{2\pi t_0}{T}\right) = x_0$$

Thus at time $t_0 + T$ it is indeed back where it was at time t_0. This is true at any time t_0: the spring will always be at the same position a time T later. This time of repeating T is called the period.

Once we know that $x = A\cos(\omega t)$, we know the velocity and acceleration as well. You can see this by drawing a picture: recall that the velocity is the slope on the displacement vs. time plot, and the acceleration is the slope on the velocity vs. time plot. Below I'll show how to compute a formula for them without using calculus. The answer is that

$$v = -A\omega\sin(\omega t)$$

$$a = -A\omega^2\cos(\omega t)$$

(If you remember your calculus, the easy way to compute these is to use $v = dx/dt$, and $a = dv/dt$.) Notice that the velocity and acceleration have the same period T.

I haven't yet told you how to find A and ω for a spring. The amplitude A depends on how you start the system: how far or the stretch is initially stretched, and what its initial velocity is. However, the frequency ω at which a spring oscillates is *independent* of the initial conditions. Since we have expressions for the velocity and acceleration, we can derive this. Put a mass m at the end of a spring and stretch it a distance x, The force on the mass is

$$F = -kx$$

(The negative sign is because it is the restoring force on the mass.) This is the only force on the mass, so Newton's second law says that

$$F = -kx = ma$$

We can plug in our expressions from above

$$-kA\cos(\omega t) = -m\omega^2 A\cos(\omega t)$$

Most of these terms cancel, leaving

$$k = m\omega^2$$

Therefore

$$\omega = \sqrt{\frac{k}{m}}$$

In other words, if you attach an object of mass m to a particular spring with spring constant k, then the frequency *must* be $\omega = \sqrt{\frac{k}{m}}$. This is completely independent of the amplitude A and any other initial conditions.

Since a pendulum obeys Hooke's Law, it also obeys simple harmonic motion. Thus θ also obeys simple harmonic motion, where the spring constant is mgL. In other words,

$$\theta = A\cos(\omega t)$$

The analog of the mass is the moment of inertia mL^2, so the frequency of oscillation obeys

$$\omega = \sqrt{\frac{mgL}{mL^2}} = \sqrt{\frac{g}{L}}$$

The longer the string is, the lower the ω, which means the longer the period (which is $\omega = 2\pi/T$). This is why grandfather clocks work – the period depends only on the length of the pendulum, and not how hard you push it to begin with.

Simple Harmonic Motion and Circular Motion

It is not a coincidence in the formula for simple harmonic motion I used the symbol ω and called it the frequency, just like we did for circular motion. Both kinds of motion are periodic with period $T = 2\pi/\omega$. In fact, simple harmonic motion is very closely related to motion on a circle. This turns out to be very useful for understanding simple harmonic motion. I'll state the relation first, and then explain it in detail.

The x component of an object moving in a circle of radius $r = A$ at constant angular speed ω obeys the relation

$$x = A\cos(\omega t)$$

Thus the x component of circular motion is the same as simple harmonic motion, where the amplitude A of the motion is the same as the radius of the circle.

You can see this explicitly by shining a light across the circle. Then put a wall on the opposite side. Thus the shadow of the object moving in a circle moves back and forth on the wall. This is one component, and it moves back and forth on the wall with the relation $A\cos\omega t$.

This correspondence is useful for finding the velocity and acceleration of the object. In fact, it allows us to find the formula for v and a without doing any calculus. Let's study circular motion in terms of the angle θ, which is where the object is on the circle. We define it so that at time $t = 0$, the object is at $\theta = 0$. The we can find theta as a position of time by using the definition

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Thus here

$$\theta = \omega t$$

This is one of our formulas from chapter 8 (the angular acceleration here is zero). To check this, note that at time $T = 2\pi/\omega$, $\theta = 2\pi$: it has indeed gone around once. Let's first check the x coordinate. At any time t, simple trigonometry gives

$$(x, y) = (r\cos\theta, r\sin\theta)$$

We've already said that for this correspondence to work, we need $\theta = \omega t$, and $r = A$. Plugging this in gives

$$(x, y) = (A\cos(\omega t), A\sin(\omega t))$$

So this proves my above claim that the x component of circular motion is simple harmonic motion.

Now let's do a little more trigonometry to get the x and y component of the velocities. They are

$$(v_x, v_y) = (-v\sin\theta, v\cos\theta)$$

To check this, notice that at $\theta = 0$ the velocity is entirely in the y direction. We can use the fact that $v = \omega r$, along with $r = A$ and $\theta = \omega t$ to get

$$(v_x, v_y) = (-A\omega\sin(\omega t), A\omega\cos(\omega t))$$

The velocity of the spring is then the x component:

$$v_{spring} = -A\omega\sin(\omega t)$$

Finally, we can get the acceleration. Since the object is moving at constant angular speed ω (there is no torque), the acceleration is all centripetal and thus directed inward. Doing a little more trigonometry gives

$$(a_x, a_y) = (-a\cos\theta, -a\sin\theta)$$

Just like we know that $v = \omega r$, we have $a = v^2/r = \omega^2 r$. Plugging this in gives

$$(a_x, a_y) = (-\omega^2 A\cos(\omega t), -\omega^2 A\sin(\omega t))$$

Thus the acceleration of the spring is the x component:

$$a_{spring} = -\omega^2 A\cos(\omega t)$$

The reason for the minus sign is that the force of the spring is a restoring force. Remember that at $t = 0$ the string was maximally stretched. Thus the force and acceleration are negative: the spring is pulling back.