Lecture 20

- Archimedes' Principle
- Incompressibility and Pascal's Principle
- Continuity Equation
- Bernoulli's Equation

Cutnell+Johnson: 11.5-11.10

Archimedes' Principle

If you've ever tried to put something hollow like a beach ball under water, you feel a very strong force pushing back up. This is called a *buoyant* force, and is a result of the difference in pressure at different heights. Consider the example we did before, only now make the container hollow. Then we have a force P_1A at the top of the cylinder, and P_2A at the bottom. Since the container is hollow, we can neglect its weight. Since $P_2 > P_1$ (due to the weight of the fluid around the container), there is a net force on the container, pushing it back up. This leads to *Archimedes' Principle*, which is very simple to state:

The buoyant force is the weight of the fluid which the object displaces.

Thus notice the buoyant force is independent of how deep you are, once the object is totally immersed. This is why ships can float, even though they're very heavy. Water is very dense, and a big ship displaces a lot of water.

Archimedes' Principle is easy to derive from the formula we derived last time. Think of a closed cylinder immersed in water. The pressure on the top of the cylinder is P_1, and the pressure on the bottom is $P_2 = P_1 + \rho gh$. This means that if the top and bottom of the cylinder each have area A, the force on the top is P_1A downward, while the force on the bottom is P_2A upwards. The net force on the cylinder is then

$$F = (P_2 - P_1)A = \rho ghA = \rho gV = mg$$

where m is the mass of the water which is diplaced by the cylinder. This is Archimdes' Principle for a cylinder; the same goes for any shape of container. The upward force is the weight of the fluid displaced.

Incompressibility and Pascal's Principle

There is one big difference between liquids and gases. The density of a gas is easy to change. However, fluids are usually *incompressible*. Incompressiblility means that the density of a fluid is independent of the pressure. In the equation we used a few lectures ago, $\Delta P = -B\frac{\Delta V}{V_0}$, incompressibility means that the bulk modulus $B = \infty$. This is not perfectly true: fluids do contract and expand a little, but not much at all: this expansion and contraction can easily be neglected. We've already used fluid incompressibility. For example, the formula for how the pressure depends on the depth of the fluid assumed that the density remained constant, even though the pressure increases. Pascal's Principle also depends on it. Pascal's principle says that if you push at one end of the fluid, the pressure increases everywhere. If the fluid were compressible, what would just happen is that part of the fluid would become more dense. This is what happens to a solid. A gas, on the other hand, will compress uniformly. Stated precisely, Pascal's principle is

> Any change in the pressure applied to a completely enclosed fluid is transmitted undiminished to all parts of the fluid and hence to the walls.

This is how a hydraulic pump works, to say lift your car up at the garage. You have a two pistons, one narrower than than the other, each pushing on the same water. The wider one supports the car. The pressure inside the fluid always obeys the formula $P_2 = P_1 + \rho gh$. Say the two pistons are at the same height. Then $P_1 = P_2$. Now say you push one piston down, thus increasing the pressure P_1. Thus P_2 increases as well. Then $P_1 = P_2$ implies that $F_1/A_1 = F_2/A_2$, where A_1 and A_2 are the areas of the two pistons. Thus

$$F_2 = F_1\frac{A_2}{A_1}$$

If $A_2 > A_1$, then the force at the second piston is larger than that at the other. This is why you put the car on the larger piston: the force you apply is magnified. If the two pistons aren't at the same height, you need to take into account the fact that the pressure at the two places is different. This you can easily do by using the formula $P_2 = P_1 + \rho gh$.

A hydraulic pump seems almost like a miracle – it can magnify a force as much as you want. Of course, you don't get something for nothing. Remember that work is force times distance. The work you do must be the same as the work done on the car. Thus if $A_2 = 10A_1$, then

$F_2 = 10F_1$. However, $W_1 = W_2$, so the distance you must push is 10 times as far as the car moves: $d_2 = d_1/10$.

Continuity Equation

So far, we've discussed static fluids. Now we'll discuss fluids in motion. If something moves, it has to go somewhere. This simple fact is just the conservation of mass (this holds unless you're in a nuclear reactor, which converts mass to energy). The conservation of mass results in what's called the continuity equation. Let's consider a small chunk of moving fluid moving at some speed v. During some small time interval Δt, this chunk moves a distance $d = v\Delta t$. The volume of fluid that has flowed past this point is therefore $V = dA = Av\Delta t$, where A is the cross-sectional area. The mass of this chunk of fluid is therefore

$$\Delta m = \rho V = \rho A v \Delta t$$

Thus this gives us the rate of mass flow:

$$\frac{\Delta m}{\Delta t} = \rho A v$$

If the fluid is in a steady state, this means mass isn't accumulating or diminishing. In other words, you're not doing something like adding a water to a jar. In a steady state, the rate of mass flow must be the same everywhere.

$$\rho A v = \text{constant}$$

Another way of writing this equation is to say that the mass flow must be the same at any two points in the same fluid. Thus

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

The continuity equation is familiar if you've ever handled a garden hose. If you shut off say half of the end with your thumb, you've decreased the cross-sectional area A in half. This means that the velocity of the water must double. The water needs to hurry up to get out. In general, this means as a tube gets wider or narrower, the fluid flow gets slower or faster. The continuity equation also applies to gases: the wind gets faster in between large buildings. Again, the point is that the mass must go somewhere. However, it is simpler for a liquid, because of the incompressibility. The density must the same everywhere, so $\rho_1 = \rho_2$. This means that

$$A_1 v_1 = A_2 v_2$$

in a liquid.

Bernoulli's Equation

Bernoulli's equation relates pressure, height and velocity. To derive it, we'll use conservation of energy. Thus we'll have to neglect ways energy can be dissipated. In fluids, the analog of friction is called viscosity. Honey is extremely viscous: when you try to stir it, most of the energy is spent on friction: the interactions between the honey molecules prevent the fluid from sliding freely past each other. On the hand, water is much less viscous, and we will neglect the viscosity altogether. This means that all the force one puts into pushing a fluid goes into fluid motion or pressure. It is simpler to row a boat in water than it is in honey. Another thing we'll assume for simplicity is the the fluid is not rotating as it flows.

With these assumptions, we can derive Bernoulli's equation. When a fluid is moving, the pressure depends on more that just the height: it depends on the velocity of the fluid as well. The reason is that for a fluid to move, you need a net force. To have a net force, you have a pressure difference. To derive Bernoulli's equation, let's first consider a pipe which goes horizontally so that the fluid isn't changing height. To make the fluid go faster, you have to do work. Thus consider two chunks of fluid, each of mass m. One has velocity v_1, and the other is down the pipe, and has accelerated to velocity v_2. To accelerate the chunk requires applying a force through a distance. In other words, it requires work. Recall in such a situation, the work is the change in kinetic energy.

$$W = \Delta KE = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

As we saw with Archimedes' Principle, the way you apply a force in a fluid is to have a difference in pressure. Thus if there is a difference in pressure between one point and another, there is a net force on the fluid in between. The pressure on the *slower* chunk needs to be *larger*, in order to acclerate it. To move the fluid a distance d requires doing the work

$$W = Fd = (P_1 - P_2)Ad = (P_1 - P_2)V$$

Equating the two says that that

$$(P_1 - P_2)V = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Rearranging the terms and using the fact that the density $\rho = m/V$, we have

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$$

This equation is useful for a Venturi meter, as in the demonstration in class.

Problem Consider two pipes with $r_1 = 2r_2$. Join them together and let water flow. The pressure difference between the two parts is $P_1 - P_2 = 100\,Pa$. What is the velocity in the larger half?

Answer First, we use the continuity equation to relate the two velocities. The two areas are related by $A_1 = 4A_2$ (note that it's 4, not 2: if the radius doubles, the area quadruples.) Thus the continuity equation says that

$$v_1 A_1 = v_2 A_2$$

so that $v_2 = 4v_1$ (the fluid in the larger pipe flows slower, and with greater pressure). Now using the above equation gives

$$\begin{aligned} P_1 - P_2 &= \frac{1}{2}\rho(v_2^2 - v_1^2) \\ 100\,Pa &= \frac{1}{2}(1000\,kg/m^3)(16v_1^2 - v_1^2) \end{aligned}$$

Solving for v_1 gives

$$v_1 = \sqrt{\frac{100}{(500)(15)}} m/s = .12 m/s$$

The velocity in the smaller section is four times as large: $v_2 = .48 m/s$.

Now we can combine the same-height equation with our earlier height-dependent equation to get the full Bernoulli's equation. We just need to include gravity like we did before, so let's consider a pipe which goes uphill. Then we need to do work to get the water to go uphill as well, from y_1 to y_2. Conservation of energy means that

$$W = \Delta KE + \Delta PE$$

(Remember the equation $\Delta KE + \Delta PE = 0$ only applies if there are no forces other than gravity, i.e. $W = 0$.) The gravitational potential energy is as always mgh. Plugging everything in gives

$$(P_1 - P_2)V = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 + mg(y_2 - y_1)$$

Rearraging again gives Bernoulli's equation:

$$P_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$

If you set the velocities equal to zero like we were doing last time, then this equation reduces to our earlier one.

There are many consequences of Bernoulli's equation. In fact, Bernoulli's equation also applies to air. It's why curve balls curve. The basic idea is that when air is moving faster, its pressure is lower. Thus if you manage to have an object like a baseball with different pressures on different sides, the object will feel a net force. A spinning baseball drags some of the air with it. This means on opposite sides, the air is moving at different speeds. This results in a change in pressure, and makes the ball move sideways.