Lecture 3

- The equations of kinematics

Cutnell+Johnson: chapter 2

The equations of kinematics

Last time we defined acceleration. This enabled us to find the equation

$$v = at + v_0. \tag{1}$$

In this equation t is what we earlier called Δt: they're the same thing; it's just easier not to write Δ every time. Equation (1) tells us how the velocity changes with time under a constant acceleration. The velocity of an accelerating object is pretty difficult to measure experimentally: you would have to have a camera which took pictures so fast that the velocity didn't change much in between each picture. An equation for the displacement is much more useful, since displacement is much easier to measure. This is what I'll derive now.

Say an object starts at velocity v_0, and undergoes a constant acceleration. After some time t, it has velocity $v = v_0 + at$ (equation (1)). Since the acceleration is constant, the velocity increases at a constant rate. Thus its *average* velocity $\overline{v}$ in this time interval is halfway between v and v_0, namely

$$\overline{v} = \frac{1}{2}(v + v_0).$$

This should be pretty obvious from the figure in the last lecture. The average velocity for any interval is the point in the middle of the interval. The average velocity is easily related to the displacement. Recall from Lecture 2 that the average velocity is defined by

$$\overline{v} = \frac{\Delta x}{\Delta t}$$

If we define position and time so that $x_0 = 0$ and $t_0 = 0$

$$\overline{v} = \frac{x}{t}.$$

We can now combine the two equations for average velocity to give

$$\frac{x}{t} = \frac{1}{2}(v + v_0).$$

Rewriting this in terms of the displacement gives the useful relation

$$x = \frac{1}{2}(v + v_0)t \tag{2}$$

What is most useful for an experiment is if we eliminate the velocity v entirely. Using equation (1), we can substitute for v, giving an even more useful relation

$$x = v_0 t + \frac{1}{2}at^2 \tag{3}$$

The equation (3) tells us how the position changes with time for an object undergoing constant acceleration. Note that if there is no acceleration, it reduces to $x = v_0 t$, familiar from the last lecture. When there is an acceleration, the t^2 means that the object is going faster and faster as time goes on. Let's do a simple

While we're at it, let's derive one more useful relation. From the equation (1), we have

$$v + v_0 = at$$

From the equation (2), we have

$$v - v_0 = \frac{2x}{t}$$

Multiplying these two gives

$$(v - v_0)(v + v_0) = at\left(\frac{2x}{t}\right)$$

Simplifying this gives

$$v^2 - v_0^2 = 2ax \tag{4}$$

The equations (1,2,3,4) are *equations of kinematics*. They relate various properties describing a body's motion to one other.

Problem Say an object is at rest, and then dropped. In terms of g, how far has the object gone after one second? Two seconds? Three seconds? This is plotted in Figure 1.

The way to calculate this is to use the formula (3). Just plug in the time:

$$\begin{aligned} d_1 &= \frac{g}{2}(1sec)^2 \\ d_2 &= \frac{g}{2}(2sec)^2 \\ d_3 &= \frac{g}{2}(3sec)^2 \end{aligned}$$

Thus from 1 to 2 seconds it travels three times as far as it traveled in the first second, because

$$\frac{d_2 - d_1}{d_1} = 3.$$

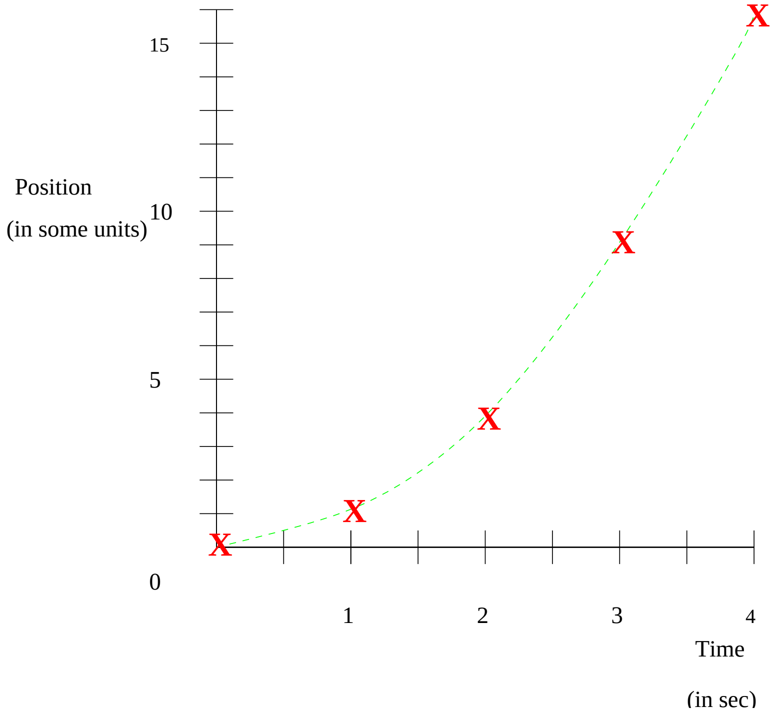

Figure 1: Positions of an object falling

From 2 to 3 seconds it travels five times as far as it traveled in the first second, because:

$$\frac{d_3 - d_2}{d_1} = 5.$$

How many times as far would it travel in the interval 3 to 4 seconds? In the interval from n to $n+1$ seconds?

The strobe light is a great way of testing this. It will even allow us to measure experimentally the value of g. The strobe light gives brief flashes of light at regular intervals. Because the flashes are so brief, a moving object moves only a tiny bit while the light is on. Thus it is frozen for that flash. While the light is off, the object moves. When the next flash comes, the object is frozen in a different position. Thus instead of seeing an object move continuously, you just see it at various fixed points, just like in Figure 1.

In the experiment we did last time, we look at milk dripping out. If we tune the strobe to be in sync with how the drops are coming out, subsequent drops will be in the same place when the strobe comes on. Thus the drops will look frozen in place, even though they are of course moving. This allows us to get an accurate measurement of where the drops are.

The strobe is a tricky way to get g accurately, it's better as a demonstration. The result for

g on the surface of the earth is (to three significant figures) is

$$g = 9.80 m/s^2.$$

As one gets farther from the earth, gravity gets weaker, and hence g gets smaller. We will study this later on in the course (chapter 5 of the book). Notice the MKS unit of acceleration is m/s^2. This looks a little funny, but you should think of it as "meters per second per second", in other words, how much a velocity changes in some time interval.

So now let's go back to the first experiment. When the weights are evenly spaced and dropped, the higher-up ones hit the ground at closer time intervals because they have more time to accelerate. **Problem** Say that blocks $2.0m$, $3.0m$ and $4.0m$ high are dropped at the same time. What are the times t_1, t_2 and t_3 they land? Is the difference $t_2 - t_1$, larger, smaller, or the same as $t_3 - t_2$?
Answer Define a coordinate system where up is the positive x direction, and set $x = 0$ to be the ground. Since $v_0 = 0$, use $x = gt^2/2$. Solve for t:

$$t = \sqrt{\frac{2x}{g}}$$

So $t_1 = 4.4$ seconds, $t_2 = 5.4$ seconds, $t_3 = 6.3$ seconds. So

$$\begin{aligned} t_2 - t_1 &= 1.0s \\ t_3 - t_2 &= .9s \end{aligned}$$

So the time interval between the higher two blocks is smaller than the time interval between the lower two.

Problem Say there are blocks $1m$ and $4m$ high. Where would you put a third block so that $t_2 - t_1 = t_3 - t_2$? Use the requirement that

$$\frac{t_3 - t_2}{t_2 - t_1} = 1$$

Then plug in $t_1 = \sqrt{2x_1/g}$, $t_2 = \sqrt{2x_2/g}$, and $t_3 = \sqrt{2x_3/g}$. All the factors of g cancel, so

$$1 = \frac{\sqrt{x_3} - \sqrt{x_2}}{\sqrt{x_2} - \sqrt{x_1}} = \frac{\sqrt{x_3} - 2}{2 - 1}$$

where x_3 is measured in meters. Solving this equation for x_3 gives $x_3 = 9m$ Thus if you place blocks at $x_1 = 1m$, $x_2 = 4m$, $x_3 = 9m$, the time interval between landings will be the same. What would x_4 be? What would x_j be for any number j?

This is the principle behind the string with weights. The first weight is placed at some distance a. The second is placed at a distance $4a$, the third at $9a$, the fourth at $16a$, and so on.

Thus the time intervals between subsequent blocks are all the same. Because of the t^2 in the formula (3), the distance between subsequent blocks must get larger and larger.

Thus we have understood how gravity works. It gives all objects in free fall a constant acceleration of $g = 9.80m/s^2$, if we neglect air resistance. We have derived several equations useful for describing the motion of objects undergoing constant acceleration.

An important thing to notice is that the acceleration can be negative, just like velocities and displacements accelerations can be negative. A negative velocity means that an object is moving to the left. What does a negative acceleration mean? It does **not** necessarily mean that the object is decelerating: deceleration means that the *speed* of the object is decreasing. Negative acceleration means that the object's *velocity* is decreasing. If the object's velocity is negative, a negative acceleration means that the velocity is becoming more negative, which means that the speed is increasing.

Whether the acceleration is negative or positive depends on how you've defined the coordinates. If you define up to be positive, then gravity acts with an acceleration $-g = -9.80m/s^2$. If you define up to be negative, the gravity acts with the acceleration $g = 9.80m/s^2$. Thus we define g to be always a positive number.

To illustrate these points, let's do a few examples. First, let's do an acceleration problem that's not gravity.
Problem A jet plane lands on a runway at 200 mph ($89.4m/s$). It immediately turns its jets in reverse, resulting in an acceleration of $-5m/s^2$. How long will it take for the plane to stop? How far will it travel in this time? What happens if the plane keeps its engines on (i.e. continues its acceleration of $-5\,m/s^2$)?

Answer First use $v = v_0 + at$ to find the time, with $v = 0$, $v_0 = 89.4m/s$, and $a = -5m/s^2$. Then

$$t = \frac{v - v_0}{a} = \frac{0 - 89.4m/s}{-5m/s^2} = 17.9 \text{ sec}$$

To find the distance, use

$$x = v_0 t + \frac{1}{2}at^2 = \left(89.4\frac{m}{s}\right)(17.9sec) + \frac{1}{2}\left(-5\frac{m}{s^2}\right)(17.9s)^2 = (1600 - 800)\,m = 800m$$

Another way to derive this distance is to have used the formula

$$x = \overline{v}t = \frac{1}{2}(v_0 + v)t = \frac{1}{2}\left(89.4\frac{m}{s}\right)(17.9sec) = 800s$$

Finally, if the plane keeps on applying this negative acceleration, it will start to go backwards!!

Another way of seeing how these minus signs matter is the example of throwing a ball up in the air. Let's pick a coordinate system where up is positive and down is negative, so the ball's initial velocity is positive and its acceleration is negative. Its acceleration at the beginning, the middle, the end, any time is always g (once the person throwing it has let go), and the direction of the acceleration is always down.

Problem I throw a ball up in the air. It remains up in the air for 3.00 seconds. What is the total displacement after I catch it? What was its initial velocity?

Answer The ball's total displacement is zero after I catch it: it ends in the same place it started. To derive its initial velocity, we can use the relation

$$x = v_0 t + \frac{1}{2}at^2 \tag{5}$$

In this coordinate system, the acceleration is $a = -g = -9.80 m/s^2$. The total displacement $x = 0$. Thus

$$v_0 t = \frac{g}{2}t^2.$$

Because this equation has a t^2 in it (a *quadratic* equation), it has two solutions. One solution is obvious: $t = 0$. This corresponds to the case where the ball is not thrown at all: $x = 0$ and $t = 0$. This is a possible physical situation, but not the one we're interested in. The other solution corresponds to

$$v_0 = \frac{g}{2}t.$$

Using the value $g = 9.80 m/s^2$ and the time elapsed of $t = 3.00$ seconds gives the initial velocity to be

$$v_0 = 14.7 m/s.$$

(which means I don't exactly have a major-league strength throwing arm: what is this speed in mph?)

This last problem is an example of the power of using vectors, which in this one-dimensional case just means allowing negative displacements, velocities and acceleration. In case that result looks a little mysterious, let's explore the problem a little more.

Problem continued How long does the ball take going up, and how long does it take to go down? What is the velocity of the ball right before I catch it? What is the total distance the ball travels on its trip?

Answer continued So to see how long it takes to go up, use the fact that the velocity is zero when the ball is at its peak. It is only zero for an instant, but it must be zero at some point, because the ball changes from positive to negative velocity: to get from positive to negative, it must go through zero. Now we can use the formula

$$v = v_0 + at \tag{6}$$

derived in the last lecture. We have $v_{peak} = 0$ at the peak, $v_0 = 14.7m/s$, and $a = -g = -9.80m/s$. Plugging this in gives us $t = 1.50$ seconds, not surprisingly. It takes half the time to go up, and half the time to go down. We use the same formula (1) to get the velocity at the end. This formula is valid for any object undergoing a constant acceleration: it doesn't matter that in the course of its motion, it goes up and then down. To get the velocity at the end, use the same v_0 and a, and here $t = 3.00$ seconds to give

$$v_{end} = 14.7m/s + (-9.80\,m/s^2)(3.00sec) = -14.7m/s.$$

Thus, as you would expect, it lands with the same speed and opposite velocity that it is thrown with (of course neglecting air resistance). Finally, to get the total distance traveled, we use the formula (3) again. To find out how far it goes on the way up, we use $v_0 = 14.7m/s$ and $t = 1.5$ seconds to get

$$x_{peak} = (14.7m/s)(1.50sec) + \frac{1}{2}(-9.80m/s^2)(1.50s)^2 = 11.0m$$

Obviously, it goes down the same distance it goes up, so the distance traveled is $2x_{peak} = 22.0m$.

Problem An object dropped out of a window falls halfway to the ground in 1.2 seconds. How long does it take to hit the ground?

Answer Call d half the distance to the ground, so that $2d$ is the total distance to the ground. For the first half we have

$$-d = -\frac{1}{2}gt_1^2$$

where $t_1 = 1.2$ seconds. For the second half, we have

$$-d = v_{1/2}t_2 - gt_2^2$$

where $v_{1/2}$ is the velocity at the halfway point. This is given by

$$v_{1/2} = v_0 + at_1 = 0 - gt_1 = -gt_1$$

Plugging this in gives

$$-d = (-gt_1)t_2 - gt_2^2$$

We now have two different equations for d. Thus

$$-\frac{1}{2}gt_1^2 = (-gt_1)t_2 - gt_2^2$$

Simplifying this gives

$$t_1^2 = 2t_1t_2 + t_2^2$$

Plugging in t_1 and using the quadratic equation gives $t_2 = .50s$, so the total time is $t_1 + t_2 = 1.7s$.