Lecture 4

- Vectors
- Motion and acceleration in two dimensions

Cutnell+Johnson: chapter 1.5-1.8, 3.1-3.3

We've done motion in one dimension. Since the world usually has three dimensions, we're going to do motion in two dimensions next. As we'll see, until we get to rotation (chapter 5), most of the physics is the same as in one dimension. The main additional complication for now will be to understand how vectors work.

Vectors

A vector is an arrow, nothing more. It is length is called the *magnitude*. The direction it's pointing is called, well, the *direction*. We've actually already done vectors in one dimension. Displacement, velocity, and acceleration are all vectors. The magnitude of the velocity is the speed; the magnitude of the displacement we usually call the distance. A magnitude is always a positive number. The direction of these one-dimensional vectors was determined by the sign. As I said in the last lecture, which direction is positive and which is negative is a matter of convention, but once you've decided, then $+$ corresponds to one direction, while $-$ corresponds to the other.

In two dimensions, the direction gives more information than just a sign. A two-dimensional direction is called an *angle*. The study of angles is called trigonometry. An angle can be labelled, for example, the way a clock works: specify which direction is 12 o'clock, and the time specifies the angle. An angle can be specified by a compass direction: specify which direction is north, then give the direction (e.g. north by northwest). Another system often uses is degrees, which divides the circle into 360 equal parts (the 360 is quite arbitrary, it must have something to do with time, since each 10 seconds on a clock make one degree, a minute is 6 degrees). We'll use degrees for now, but later in the course we'll use what are called radians: there are 2π of them in a circle.

Specifying a vector by its magnitude and direction is very useful, quantitatively and qualitatively. However, to see the physics clearly it is often necessary to describe a vector in terms of its *components*. Notice that before we described a two-dimensional vector using two numbers: magnitude and angle. Specifying the angle requires defining a single reference direction (e.g. north, 12 o'clock, etc.). One then measures angles with respect to this single direction. Measuring the components of a vector requires specifying two reference directions, which we usually call x and y. We usually specify up to be y, and the horizontal direction to be x. One then describes the vector by its x and y components. This is done by placing the beginning of the vector at the origin of the coordinates. Then the end lies at some point (x,y). This point specifies the vector uniquely. Note that this way of specifying a two-dimensional vector also requires two numbers.

When I use vectors, I will label them with an arrow, i.e. $\vec{r}$ is the vector which describes position in two dimensions. Your book uses boldface, but it is hard to do boldface on the blackboard, and I will use the arrow in the notes. In components, $\vec{r} = (x, y)$. Velocity and acceleration are usually labeled $\vec{v} = (v_x, v_y)$ and $\vec{a} = (a_x, a_y)$. In this notation, the magnitude is labeled as e.g. $|\vec{r}|$, or if there's no ambiguity, just plain r (the vector without the vector sign, or in the book, without the boldface).

The magnitude of a vector can easily be obtained from the coordinates. Using the Pythagorean theorem gives

$$\text{magnitude} = |\vec{r}| = r = \sqrt{x^2 + y^2}.$$

To find the angle requires trigonometry; one has

$$\tan\theta = x/y$$

if we measure the angle θ from the x-axis. One has also

$$x = r\cos\theta \qquad y = r\sin\theta.$$

If you multiply a vector by -1, it corresponds to multiplying each component:

$$-\vec{r} = (-x, -y)$$

This corresponds to the opposite of the original direction.

One frequently needs to add vectors together. It is very important to recognize how vector addition works. We've already seen how it works in the problem of throwing a ball into the air: the displacement on the way up is some vector y, while the the displacement on the way down is $-y$. Adding them together one gets the total displacement $y + (-y) = 0$. Notice we have not just added the magnitudes, but rather the full one-dimensional vector. In other words, it was vital that we included the signs. In two dimensions, this gets more obvious. Say I move 4 miles east, then 4 miles north. How far am I from where I started? Clearly it is not 8 miles. To calculate this requires *vector addition*. The sum of two vectors is easy to do in components: just add the components *individually*: thus the vector (x_1, y_1) added to (x_2, y_2) is $(x_1 + x_2, y_1 + y_2)$. The vector for moving east is (4 miles, 0), while for moving north is (0, 4 miles). Adding them together gives a total displacement of (4 miles, 4 miles). The distance from where I started is the magnitude of this displacement, which is $\sqrt{32}$ miles≈ 5.65 miles. That one is sort of obvious, but it gets trickier. How far have I traveled if I travel 4 miles northeast, then 4 miles north?

Motion in Two Dimensions

There is one key fact you must understand to do motion and forces in dimensions higher than one. The fact is simple:

> All the rules we learned in one dimension apply to any **component** of the displacement, velocity, acceleration,. . . in higher dimensions.

This sounds obvious, but it has many interesting consequences. For example, gravity in the absence of other forces *does not affect horizontal motion at all.* This also means that *horizontal motion does not affect the vertical acceleration at all.* Say we drop a ball out of a airplane moving at constant velocity. Then we can divide the problem into a vertical part and a horizontal part, and study them separately. Both of these problems are ones we've studied before, and simple ones at that. The vertical part simply consists of a ball in free fall. The horizontal part consists simply consists of a ball moving forward at a constant velocity. So let's do a

Problem I drop a ball out of a plane moving at constant velocity $200m/s$ at constant altitude $5000m$. How long does it take to fall? What is its velocity when it hits the ground? How far is the point it lands at from the point I drop it at?

Answer First let's set up the problem in vector components. The initial position of the particle is $(0, 5000m)$. The initial velocity of the particle is solely in the x-direction, so

$$\vec{v}_0 = (200m/s, 0).$$

(The reason the ball is moving horizontally initially is because the plane is.) The only acceleration in the problem comes from gravity. Gravity acting alone pull things down, which here is the y direction. Thus the acceleration is constant and is

$$\vec{a} = (0, -9.8m/s^2).$$

Because each component is its own individual problem,

- The horizontal motion does not change the vertical acceleration at all, so the time it takes to fall is the same as if the plane were not moving.

- The vertical acceleration does not change the horizontal motion at all, so the ball just continues at the same v_x throughout.

Let's first find out how long it takes to fall. This is determined by the y components. We'll use the equation

$$y = v_{0y}t + \frac{1}{2}a_y t^2.$$

This equation should be familiar to you by now; we've already used it many times. We've written it here to describe the motion in the y direction, so that's why I've replaced the displacement, initial velocity, and acceleration by their y components. The time t is not a vector, so we just keep that t. This is just a ball in free fall. It starts out at zero velocity, because $v_{0y} = 0$. Thus the time to fall is given by

$$-5000m = 0 + \frac{1}{2}(-9.8m/s^2)t^2,$$

so $t = 32s$. Then we can find the vertical component of the velocity, using the formula

$$\begin{aligned} v_y &= v_{0y}t + at \\ &= 0 + (-9.8m/s^2)(32s) \\ &= -310m/s \end{aligned}$$

This is just the y-component of the velocity. The x component is easy, however. Since there are no forces in the x direction, there is no acceleration (we neglect air friction). Thus the velocity does not change in the x direction: it stays at $200m/s$. Thus the final velocity is then

$$\vec{v} = (-310m/s, 200m/s).$$

Finally, we need to figure out how far its landing point is from the initial point. Obviously, the displacement in the y direction is $-5000m$. It is easy to find the displacement in the x direction as well, by using the formula

$$\begin{aligned} x &= v_{0x}t \\ &= (200m/s)(32s) \\ &= 6400m \end{aligned}$$

Thus if its initial position is $\vec{r}_0 = (0, 5000m)$, its final position is $\vec{r}_f = (6400m, 0)$. The total displacement is then the difference of the two:

$$\Delta\vec{r} = \vec{r}_f - \vec{r}_0 = (6400m, -5000m)$$

(Note this is a problem where we chose $\vec{r}_0 \neq 0$, but of course the displacement (since it is the change in position) is independent of this choice. The distance between the two points is the magnitude of the displacement, which is

$$\Delta r = \sqrt{(5000m)^2 + (6400m)^2} = 8100m$$

I've used the convention that Δr is the magnitude of $\Delta \vec{r}$. This does not mean the ball's path is $8100m$ long: the ball does not travel in a straight line, it travels in an arc.

To recap this somewhat long problem: the displacement, velocity and acceleration equations we derived for one dimensional motion apply to each component of the two-dimensional problem. All of the individual steps are pretty easy (assuming you understand chapter 2, that is). The hard part is separating all the different effects from each other.

One thing also to note from the last problem is that the ball lands right underneath the plane (as long as the plane does not divert its course after dropping the ball). In other words, the plane's position at $t = 32s$ is at $(6400m, 5000m)$, $5000m$ above the ball, which is at $(6400m, 0)$. This is a matter of some importance if you're dropping the old-fashioned dumb bombs: you don't really want the bomb right under you. Thus after dropping the bombs, the planes usually turn as fast as they can.