Lecture 8

- Rotational Motion
- Centripetal Acceleration
- Centripetal Force

Cutnell+Johnson: sections 5.1-5.4, 5.6-5.7

Rotational Motion

We're all familiar with rotational motion. CD and record players go around a central spindle, satellites orbit the earth, the earth orbits the sun. The *period* T of a circular motion is defined as the time it takes to go around once (i.e. complete one orbit). Thus the period of the earth's rotation around its axis is one day, the period of the earth's rotation about the sun is one year, etc. Often the rotation is described in terms of the rate of revolution, which is just $1/T$. In other words, a turntable of my youth revolves at 33 1/3 revolutions per minute (rpm), so its period is 1/(33 1/3) min= .03 min= 1.8 seconds.

One of the tricky things about rotation is that for a rotating object, the speed (as measured in distance/time) depends on how far it is from the center of rotation. The reason is that the distance it travels in a given orbit depends on how far it is from the center: the farther it is from the center, the farther the object must travel to make it around one trip. A circle has length $2\pi r$, where r is the radius of the circle. Thus in a given cycle, the object has traveled a distance $2\pi r$ in a time T, so the speed is

$$v = \frac{2\pi r}{T}$$

Problem The earth is $1.50 \times 10^{11}\ m$ (93 million miles) from the sun. What is its speed in m/s (neglecting the motion of the sun through the galaxy)?

Answer

$$\begin{aligned} v &= \frac{2\pi 1.5 \times 10^{11}\, m}{1 \text{ year}} \frac{1 \text{ year}}{3.65 \times 10^2 \text{ days}} \frac{1 \text{ day}}{24 \text{ hours}} \frac{1 \text{ hour}}{3.60 \times 10^3 \text{ days}} \\ &= 2.99 \times 10^4 m/s \end{aligned}$$

I should also note that the earth's orbit is not exactly a circle, but rather an ellipse (with the sun at one of the foci, if you know what that is). Here and elsewhere, we'll just assume it's a circle.

That one was too easy. We can make it trickier by looking at CDs. CDs spin around just like old-fashioned turntables. However, there are many big differences between a CD player and my beloved old turntable, even at the level of how they spin around. In a turntable the period and thus the rpm are of course independent of where the needle happens to be. Since the linear speed is faster on the outside, this means there is "more music" on the outside. CDs are different. The motor speeds up or slows down so that the *speed* is the same, no matter where the laser (the CD equivalent of the needle) is at. Thus the 1's and 0's on the CD are read by the laser at the same rate, no matter if you're playing a track on the inside or the outside.

Problem A CD rotates, roughly, at angular velocities between 300 and 600 revolutions per minute, depending on where the laser is. When the laser is $5\,cm$ out, the CD goes through 6 rotations per second. across. Where is the laser when the CD is going through 12 rotations a second?

Answer. The period T and the rpm $1/T$ change so that the speed v measured in say cm/s is the same no matter where you are. So let's calculate it on the outside. At 5 cm from the center, the distance it travels in one revolution is $10\pi\ cm$. The angular speed is the slowest on the outside, when the laser is on the outside, it takes 1/300 min = .2 seconds to go around. Thus

$$v = \frac{10\pi\ cm}{.2\ s} = 50\pi cm/s$$

When the laser is at the innermost point, the motor speeds up to 500 revolutions per minute in order to preserve the same velocity $130m/s$. The period here is 1/600 min= .1 seconds. Here we can use $v = 2\pi r/T$ find r_{inner}:

$$50\pi cm/s = \frac{2\pi r_{inner}}{.1s}$$

So $r_{inner} = 2.5cm$. By the way, a CD player plays from the inside to the outside, not from the outside in.

Centripetal Acceleration

One other tricky thing about circular motion is that even though the speed may remain the same, the velocity must change. The reason is that the direction of the motion is constantly changing. The object rotating must always be turning. This is an important thing which can happen in two dimensions. The speed (the magnitude of the velocity vector) remains the same, but the velocity vector itself changes in time. This is obvious if you just draw the arrows.

What's not so obvious perhaps is that if the velocity changes, then there must be an acceleration, even if the speed is not changing. There's no way around this. Recall that the acceleration is defined as the change in velocity over the change in time. If the velocity changes, the acceleration is not zero. What's even trickier is the direction of the acceleration. Here's one place where our everyday experience can lead to the wrong answer. If you're in car going around a curve, you feel as if you're being thrown to the outside. Thus most people conclude that since you feel this way, the force must be outward. This "force" even has a name: centrifugal force. However, this is not a force: this is just Newton's First Law! You're trying to go in a straight line (inertia), it's the car that's turning. Thus the force on you is not throwing you out. The force is actually the door or the seat belt on you, pulling you in. This force prevents you from flying out of the car in the straight line you'd like to go in. This force has a name too. It's called *centripetal force*. This is the force which forces you to move in a circle: the force which prevents you from flying out of the car.

"Centrifugal force" is not really a force. It only looks like a force from the point of view of the car, where it seems as if you're mysteriously moving outward. To a bird flying above, it's obvious that the force is towards the inside.

This direction is apparent if you also start drawing the arrows. If you draw the velocity arrow at one time, and then a short time after, for it to remain on the circle the velocity vector is moving towards the *inside*. Since the *change* in velocity is towards the inside, the acceleration is towards the inside.

Another way of thinking about this is to realize that if you attach a rock to a string and swing it around, you need to *pull* on the string to make the rock go around. Pulling it corresponds to a force pointing in.

So if we've got an acceleration in, how large is it? To compute it, first draw the two velocity vectors. After some short time Δt, the velocity vectors remain the same length, but are rotated from each other by some angle $\Delta\theta$ (the larger Δt is, the larger the $\Delta\theta$). The difference between these two vectors is the change in velocity $\Delta\vec{v}$. To compute the magnitude Δv, we just a trick. Namely, let's consider the position vector $\vec{r}$, which is just sweeping out a circle like the velocity. Thus after a time Δt, the vector $\vec{r}$ is the same length but rotated by the *same* angle θ. Drawing a picture makes it clear therefore that

$$\frac{\Delta v}{v} = \frac{\Delta r}{r}$$

But Δr (the magnitude of $\Delta\vec{r}$) is easy to compute. In a small time Δt, the object travels a distance $v\Delta t$ (as long as Δt is small, we can neglect the fact that it moves in an arc, not a straight line). Thus

$$\frac{\Delta v}{v} = \frac{v\Delta t}{r}$$

The magnitude of the acceleration is then

$$a = \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

The direction, as I explained above, is toward the center.

So we have derived a very useful fact: the total acceleration of an object in circular motion of radius r at velocity v is of magnitude v^2/r.

Centripetal Force

Since an object in circular motion is accelerating, one must apply a force to cause this acceleration. (Remember the second law: if there is an acceleration, then there must be a net force).

Problem The earth rotates the sun because of the attraction of gravity from the sun. The earth's mass is $6.0 \times 10^{24}\,kg$. What is the force the sun needs to apply onto the earth to keep the earth in orbit?

Answer We calculated above that the earth's velocity of orbit is $3.0 \times 10^4 m/s$, and used the fact that the distance from the sun is $1.5 \times 10^{11} m$. Thus the acceleration of the earth's orbit is

$$a = \frac{(3.0 \times 10^4 m/s)^2}{1.5 \times 10^{11} m} = 6.0 \times 10^{-3} m/s^2$$

So that's a very small acceleration, compared to say the effect of gravity on the earth (this is why you don't have to worry about the sun's gravity in the problems we did earlier). The force on the earth necessary to cause this acceleration is

$$F = ma = (6.0 \times 10^{24}\,kg)(6.0 \times 10^{-3}\,m/s^2) = 3.6 \times 10^{22} N$$

Since the earth is so large, the force must be large, even for such a small acceleration.

It is important to remember that the acceleration is related to the sum of the forces. In the above problem, gravity was the only force. Let's do a problem where there is more than one force.

Problem I'm spinning a yo-yo in a *vertical* circle of radius $.5m$ and at a constant angular speed of 2 revolutions per second. The yo-yo weighs $.1\,kg$. What is the tension in the string at the top of the circle? at the bottom?

Answer It's important to note that the centripetal acceleration is related to the *total* force. It's not a force in and of itself the forces in this problem are two we've already dealt with: tension and gravity. It travels a distance $2\pi r$ in one revolution, so its speed is

$$v = 2\pi(.5)\frac{m}{\text{revolution}}2\frac{\text{revolution}}{s} = 2\pi m/s$$

The fact that the yo-yo is moving in a circle means that the *total* acceleration is of magnitude

$$a_{total} = \frac{v^2}{r} = \frac{(2\pi m/s)^2}{.5m} = 80m/s^2$$

The direction of this centripetal acceleration is always towards the center of the circle. Thus the net force is of magnitude

$$\Sigma F = ma_{total} = (.1\,kg)(80m/s^2) = 8N$$

pointing towards the center of the circle (where I'm holding the yo-yo). Now let's draw a force diagram. At the top of the circle, we have gravity down and tension down. At the bottom of the circle, we have gravity down and tension up. At the bottom of the circle, therefore have

$$\Sigma F = T_{bottom} - mg$$

At the top of the circle, we have

$$-\Sigma F = -T_{top} - mg$$

The reason for the minus signs is that all of these forces are pointing down. Notice how at the top, the tension and gravity are in the same direction, while at the bottom they oppose each other. Now we can solve for the tensions. At the bottom, we have

$$8\,N = T_{bottom} - (.1\,kg)(10m/s^2)$$

so $T_{bottom} = 9\,N$. At the top, we have

$$-8\,N = -T_{bottom} - (.1\,kg)(10m/s^2)$$

so the magnitude $T_{bottom} = 7\,N$. Thus I have to put more force on it on the bottom at the top to keep it going in a circle.

To reiterate what I said above: the purpose of this centripetal acceleration is to keep the object from flying off.

Banked Curves

A banked curve is a nice example of centripetal acceleration, as well of some of the things we did in the previous chapter. First let's draw a force diagram for a car going around a curve which is not banked. To make the car go in a curve, we need a net acceleration inwards. Since gravity goes up and the normal goes down, the only force which can point inwards is friction. Thus only friction can make the car go in a circle, and of course if there's an oil slick or some ice, friction doesn't work too well. The faster you're going, the larger the acceleration must be (in fact, notice it goes up as v^2). Thus you need to slow down substantially to go around a curve.

The idea behind a banked curve is to make the normal force work as a centripetal force. If the curve is tilted upward, then the normal force is pointing (partially) inward. Thus it aids friction, and since the amount of ice does not change the normal force, can be very helpful.

Problem Assume the road is frictionless. For a given velocity v and radius of curvature r, how much do you need to bank the curve to keep the car on?

Answer Let's look at the force diagram. The gravity points down, and the normal points perpendicular to the surface. Those are the only forces in the problem. Now we need to consider what the net acceleration and net forces are. This is where the problem differs from the inclined plane problems we did before. There, the acceleration (if any) was down the plane. Here we want the acceleration to be pointing horizontally, so that the car goes in a circle. We don't want any vertical acceleration (that would correspond to the car flying into the air). Thus the forces in the vertical direction must cancel. This means that

$$\Sigma F_y = F_N \cos\theta - mg$$

This fixes the normal force. The net force in the horizontal direction is

$$\Sigma F_x = F_N \sin\theta$$

This is directed inward (in the x direction). The centripetal acceleration we need for the car not to slide up or down the curve is v^2/r, so the net force inward must be

$$\Sigma F_x = mv^2/r$$

This means that $F_N \sin\theta = mv^2/r$, so

$$\tan\theta = \frac{v^2}{rg}$$

If the angle is too small, the car slides up the bank, if the angle is too large, it slides down the curve (remember there is no friction).