Modern Physics notes – Spring 2007
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Lecture 11

- Bosons and fermions

- Feynman, 4.1-4.3, 4.7

Bosons and fermions

Electrons are indistinguishable: if you scatter Alice alpha particle off of Bob alpha particle, there's no way any detector afterwards can tell if it's Alice or Bob going through. Thus such particles are called *identical*.

There are two types of identical particles. With one type, we add the two amplitudes in the scattering process discussed last time. With the other type, we subtract the two. Let me go through this in terms of the bra and ket notation we've introduced. Remember that we have

$$f(\theta) \equiv \langle \theta, \pi - \theta | s_1, s_2 \rangle$$

The actual amplitude $f(\theta)$ will depend on the details of the experiment, but we won't need its detailed form here. When the particles are distinguishable, the probability this scattering event happens (i.e. particle 1 ends up at angle θ, particle 2 ends up at $\pi - \theta$) is $|f(\theta)|^2$. Likewise, the probability that particle 1 ends up at angle $\pi - \theta$ and particle 2 ends up at angle θ is $|f(\pi - \theta)|^2$. This means that the amplitude for this is

$$\langle \pi - \theta, \theta | s_1, s_2 \rangle = e^{i\delta} f(\pi - \theta)$$

There can be an arbitrary phase $e^{i\delta}$ here, because it does not affect the magnitude squared.

When the two particles being scattered are identical, we can't tell the difference between the two final states. This phase δ does matter when we add the amplitudes. In fact, this is the interference! The probability for the identical particles is

$$|\langle \pi - \theta, \theta | s_1, s_2 \rangle + \langle \theta, \pi - \theta | s_1, s_2 \rangle|^2 = |f(\theta) + e^{i\delta} f(\theta - \pi)|^2$$

The point is that in the process of exchanging the two particles, we must multiply the state (and hence the amplitude) by the phase. In our notation, we have

$$\langle \pi - \theta, \theta | = e^{i\delta} \langle \theta, \pi - \theta |$$

Since if we exchange the two again, then we must multiply by $e^{i\delta}$ again. Since we're back to where we started, we must end up with $e^{2i\delta} = 1$, so the phase $e^{i\delta} = \pm 1$. Thus δ can only be 0 or π. These are the two cases we saw in the scattering experiment.

If we rerun this scattering experiment again and again, we find that this relative phase is an *intrinsic* property of the particles. This means that some particles have a $+$ sign, some have a $-$ sign, but exchanging two of a given type of particle will always result in the same sign. Electrons, protons and neutrons always have the $-$ sign. Particles of this type are called *fermions.* Particles with a $+$ sign are called *bosons.* A photon is a boson. Writing this in terms of states, for two identical fermions f_1 and f_2, one has

$$|f_2, f_1\rangle = -|f_1, f_2\rangle$$

For two bosons b_1 and b_2, one has

$$|b_2, b_1\rangle = |b_1, b_2\rangle$$

This fits in well with the indistinguishability: if Alice electron and Bob electron behaved with different signs, then they would be distinguishable. This property of a $\pm$ sign under exchange is called the *statistics* of the particle. It turns out, for not-at-all obvious reasons, the fact of whether a given particle is a boson or fermion is related to its *spin*, a fact we will discuss a week or two down the road. By the way, the fact that $e^{i\delta}$ can only be ± 1 (i.e. that the statistics can be only bosonic or fermionic)is true in a three-dimensional world, but more complicated and interesting kinds of statistics occur in two dimensions. Some of these are being exploited currently in an attempt to build a quantum computer (more on this later in the class!)

Note that if you bind two fermions together (so that they cannot be split), you get a boson. For example, an alpha particle is two protons and two neutrons bound together in the nucleus. If you exchange two alpha particles, you get four minus signs and so a plus sign. Thus an alpha particle is a boson. Not all nuclei are bosons: there aren't always the same number of neutrons and protons (e.g. hydrogen). A hydrogen atom is a boson, but a ^3He atom is a fermion: there are two protons, one neutron, and two electrons. The fact that it is a fermion means that at very low temperatures it behaves completely differently from ^4He, with all sorts of interesting and strange phenomena.

The Pauli Exclusion Principle

In the scattering of identical particles I discussed last time, we saw that there were effectively two "paths" to the same outcome. By the two "paths" here, I don't mean the two individual paths, I mean the two ways the system has to get from the initial state $|s_1, s_2\rangle$ to the final state. The point is that for identical particles, the final states $\langle\theta, \pi - \theta|$ and $\langle\pi - \theta, \theta|$ are indistinguishable. We must thus take a linear combination of the *amplitudes* of the two process, just like in the double-slit experiment.

When the identical particles are fermions, one must take the linear combination $f(\theta)-f(\pi-\theta)$ with a relative minus sign. This is probably the most important minus sign in physics. We saw already one effect of the minus sign: the fact that the scattering of two identical fermions at $\theta = \pi/2$ vanishes. But the effect of this minus sign goes much deeper than this. One major consequence is called the Pauli exclusion principle. It says simply that

Two identical fermions cannot be in the same state.

The same state implies *everything* is the same: the spin, the direction, the position, etc. When you combine this fact with the fact that the values of energy, momentum, angular momentum, are quantized, the effects of the exclusion principle are enormous.

The Pauli exclusion principle for fermions explains why atoms and molecules have the structure they do. If not for it, chemistry (and hence our universe) would be completely different. A general principle of physics is that a system is stable when everything is in its lowest-energy state. Applying this principle to an atom, one might guess that all the electrons should be in the level with lowest energy. Instead, one finds only one. The reason that you can have more than one electron in a given energy level in an atom is that for the Pauli exclusion principle to hold, the electrons must be truly identical. In an atom with more than one electron, electrons **must** be in higher energy levels. The different chemistry of the elements in the periodic table comes from the number of electrons in the highest-energy levels. This explains why elements in the same *column* have similar chemical properties despite a very different number of electrons: elements in the columns have the same number of electrons in their highest energy levels.

To fully understand the structure of the periodic table, we need to understand the quantization of angular momentum and spin. But that's a long story, and before we do that, we'll first look at properties of bosons.