Modern Physics notes – Spring 2007
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Lecture 12

- Normalization of the probability amplitude
- Enhanced scattering for bosons
- Emission and Absorption of Photons

- Feynman, 4.1-4.4

Normalization

A few lectures back, I discussed how to normalize the probability amplitude. Let me review that here, because we'll need it to explain bosons in more depth. The idea is simple: in any experiment, the probabilities of all the possible outcomes must add up to 1. For example, by definition the particle in the box has probability 1 to be somewhere in the box. To see this how to describe this mathematically, we need to define the amplitude more precisely than we have before. Although most people (including me) say that the magnitude squared $|\phi(x)|^2$ is the probability for the particle to be at x, to define this precisely we must consider the probality for the particle to be in a very small region centered at x. (The reason is that strictly speaking, the probability for a particle to be at any single point must be zero, since there are an uncountably infinite number of points on the line.) Let's take this region to have a length dx, or in 3d a volume d^3x. Then, by definition the probability the particle is in this small region is

$$|\phi(x)|^2 d^3x$$

Then to get the probability that the particle be somewhere in the universe, we sum up the probabilities over all the regions. Mathematicians have a nice symbol for this: it looks like

$$\int_{\text{universe}} |\phi(x)|^2 d^3x = 1$$

Doing an integral over the whole universe is tricky (but often not impossible!) so we often simplify problems by requiring that the particle be in some specified region. Thus for the particle in a box, we have

$$\int_0^L |\phi(x)|^2 dx = 1$$

Using this condition to find the overall constant in front of a function is usually called "fixing the normalization". For example, for the particle in a box, this enabled us to fix the coefficient A in $\phi_{\text{box}}(x) = 2Ai\sin(px/\hbar)$.

Enhanced scattering for bosons

The difference between fermions and bosons can be understood roughly in the two statements "identical fermions don't like each other", and "identical bosons like each other". The more precise statement of the former is in the Pauli exclusion principle. An experimental manifestation of the latter is in a laser. A laser consists of many photons, all in the same quantum state (the currently-fashionable buzzword for this is "coherence"). It is possible to get a huge number of photons in the same quantum state because photons are bosons.

In fact, identical bosons prefer to be in the same state as each other. To illustrate this further, we'll consider a situation where we have a fixed scatterer (think of it as a rock), and a bunch of particles $a, b, c \ldots$ scattering into directions $1, 2, 3, \ldots$. We have to be careful about what we mean by a direction and a state. By direction 1 we mean that this is a small area we call dS_1. Think of this physically as the area of the front of the detector. We assume the area dS_1 is small enough so that the amplitude doesn't vary appreciably across the area. Then we normalize the states so that

$$|\langle 1|a\rangle|^2 dS_1$$

is the probability that particle a scatters off the rock into the detector 1 with area dS_1. As above, to get the probability the particle enters some region of area ΔS, we integrate:

$$\int_{\Delta S} |\langle 1|a\rangle|^2 dS_1$$

Likewise, the probability particle b scatters into some other area dS_2 is $|\langle 2|b\rangle|^2 dS_2$, and so on for all the particles.

Now let's consider the situation where both particles are present, and figure out the probability particle a scatters into dS_1 and b scatters into dS_2. If the particles are distinguishable, then this is just the product of the two:

$$|a_1 b_2|^2 dS_1 dS_2$$

where I've simplified the notation by defining $a_1 = \langle 1|a\rangle$, etc. To simplify things a little, let's assume that a and b don't vary over the entire larger region ΔS, so that $a_1 = a_2 \equiv a$ and $b_1 = b_2 \equiv b$. Then the probability $P_2(\Delta S)$ both particles will land in the region ΔS is

$$P_2(\text{distinguishable}) = |a|^2 |b|^2 (\Delta S)^2$$

If the particles are identical, we don't know whether it's particle a going into dS_1, or particle b. Thus we have probability

$$|a_1 b_2 + a_2 b_1|^2 dS_1 dS_2$$

So to get the full amplitude, we integrate over both dS_1 and dS_2. But there's a subtlety. If we were to let both dS_1 and dS_2 both range over the full ΔS, we would overcount by a factor of 2.

We can't distinguish between 1) particle a in region 1 and particle b in region 2 and 2) particle b in region 1 and particle a in region 2, we shouldn't integrate over it twice. Thus if we do the full integral, we need to *divide* by a factor of 2. Again, let's assume that the amplitude doesn't vary much over some larger region ΔS. Then the probability is

$$P_2(\text{Bose}) = \frac{1}{2}\left(4|a|^2|b|^2\right)(\Delta S)^2 = 2|a|^2|b|^2(\Delta S)^2$$

So the total enhancement is just a factor of 2, not a factor of 4.

We can now repeat this for n particles $a, b, c, \ldots$ going into regions $1, 2, 3, \ldots$. If they are distinguishable the probability will be

$$= |a_1 b_2 c_3 \ldots|^2 dS_1\, dS_2 \ldots dS_n$$

If they don't vary much over the region ΔS, then

$$P_n(\text{distinguishable}) = |a|^2|b|^2|c|^2 \ldots (\Delta S)^n$$

If they are identical particles we must add the *amplitudes* as before, not just multiply them. There are $n! \equiv n \cdot n - 1 \ldots 2 \cdot 1$ different possibilities for the scattering, e.g. for 3 particles we have $a \to 1, b \to 2, c \to 3$; $a \to 2, b \to 1, c \to 3$, etc. Then we have

$$|a_1 b_2 c_3 + a_2 b_1 c_3 + \ldots|^2 dS_1\, dS_2\, \ldots dS_n$$

If the individual amplitudes don't vary much over this region, we then have $a_1 = a_2 = a_3 = \ldots a$, etc. We have the same subtlety this time: we overcount by a factor of $n!$ by not being able to distinguish between the particles. The total probability of finding all n particles in the region ΔS is then

$$\begin{aligned} P_n(\text{Bose}) &= \frac{1}{n!}|n! abc \ldots|^2 (\Delta S)^n \\ &= n! |abc \ldots|^2 (\Delta S)^n \\ &= n! P_n(\text{distinguishable}) \end{aligned}$$

Thus the enhancement due to having identical bosons is huge if there are many particles. $n!$ grows even faster than an exponential: for large n, $n! \approx n^n$.

Emission and absorption of photons

We developed these rules for scattering experiments where the final number of particles is the same as the initial. However, bosonic enhancement is are much more general than that. We know that photons can be created (when an electron falls from an excited state in an atom) and annihilated (in the photoelectric effect).

One way to see this enhancement is to look at the difference between P_n and P_{n+1}. Assume there are n particles present, and are asking for the probability another particle will be in the same state. Our formula from above shows that

$$P_{n+1}(\text{Bose}) = (n+1)|a|^2 \Delta S \, P_n(\text{Bose})$$

where $|a|^2 \Delta S$ is the probability of getting the particle into a detector of area ΔS if there are no other particles present. If the particles were distinguishable, we have

$$P_{n+1}(\text{distinguishable}) = |a|^2 \Delta S \, P_n(\text{distinguishable})$$

Note the extra factor of $n+1$ if they are bosons: if there are n particles already present, then the probability is enhanced by a factor of $n+1$. Just the presence of the other particles makes it more likely that the boson will get into the detector.

To illustrate this, let's study creation of photons in an atom (the reverse of the photoelectic effect. Say there is an atom which can absorb and emit photons of frequency ν, i.e. two of the atom's energy levels are separated by the energy $h\nu$. Let

$$\langle \text{lower} + \text{photon} | \text{excited} \rangle = a$$

be the amplitude for the electron to fall to a lower-energy state and emit a photon, if there are no other photons present. This means that $|a|^2$ is the probability of this transition for the atom without the photons.

Now consider a situation where you have a bunch of these atoms, and the photons they emit, trapped in a box. Consider n photons trapped in the box, all in the same state. The "same state" means that the n photons all have the same polarization and frequency. In other words, these n photons are indistinguishable. The process of emitting a photon into this state is just like scattering $n+1$ photons into the detector. Because of the presence of the other photons, the probability the atom will emit a photon is enhanced. Namely, when there are n photons already present, the probability of emitting another one is not just $|a|^2$, but is enhanced to $(n+1)|a|^2$. In terms of the amplitude, we have

$$\langle \text{lower}, n+1 \text{ photons} | \text{excited}, n \text{ photons} \rangle = \sqrt{n+1}a$$

You might naively think that the probability to annihilate a photon then would be discouraged by the presence of others. That's wrong: the probability to annihilate is enhanced too. This is a consequence of a symmetry called "time reversal". Remember we define $\langle f|i \rangle$ as the probability amplitude for some initial state $|i\rangle$ to evolve to state $\langle f|$. Say we reverse the arrow of time, and ask for the amplitude to go from $|f\rangle$ to $\langle i|$. The consequence of time-reversal symmetry is that the two amplitudes are complex conjugates of each other:

$$\langle i|f \rangle = \langle f|i \rangle^*$$

(A slight comment: time-reversal symmetry can be violated in physical systems, but this relation still holds.) Applying this to our situation means that if you start with $n + 1$ photons present, the amplitude for annihilating 1 is

$$\langle n|n+1\rangle = \langle n+1|n\rangle^* = \sqrt{n+1}a^*$$

Thus photons have an enhanced probability to annihilate as well!