Modern Physics notes – Spring 2007
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Lecture 14

- Black-body radiation

- Feynman, 4.3-4.6

Black-body radiation

Last time we derived a formula for the number of photons of a given frequency at a temperature T. We now want to know how the energy in the box depends on the frequency. We first assume for every allowed photon frequency there is some atom in the box or in the walls which can create a photon at that frequency. (This won't happen for isolated atoms, but when atoms are close enough together, they interact, and you do indeed get fairly white light.)

We now want to know how much energy is in light in the frequency interval between ω and $\omega + \Delta\omega$. (Because the book uses this notation, I'm going to use "angular frequency" $w = 2\pi\nu$ instead of frequency ν. Now the energy of a given photon is $E = h\nu = \hbar\omega$. There's no major reason for doing this, but it's conventional.) We know how many photons have a given frequency ω from the Bose-Einstein distribution, but what we don't know yet how many allowed frequencies there are in this frequency interval. As we saw long ago, which frequencies are allowed depends on the size of our box.

First, let's do the one-dimensional case. We've already discussed how to see which modes are allowed in a box. We require that there be standing waves in the box. In a one-dimensional box of length L, this meant that the wavelength must be $2L/j$, where j is an integer. In terms of $k = 2\pi/\lambda$, this is $k = \pi j/L$. Let $\Delta\mathcal{N}$ be the number of modes in the range $\omega, \omega + \Delta\omega$. We have $\omega = kc$, so we know the separation in frequencies between successive modes is $\pi c/L$. This means that

$$\Delta\mathcal{N} = \frac{\Delta\omega}{c\pi/L} = \frac{L}{c\pi}\Delta\omega$$

It makes sense that this increases with L: the bigger the box, the separation between frequencies goes down, so there are *more* frequencies in a given range.

This can be generalized to a three-dimensional rectangular box of dimensions L_x by L_y by L_z. We saw that waves in three dimensions have amplitudes with an $e^{i\vec{k}\cdot\vec{x}}$. Demanding that the wave vanishes at the edges of the box means that in the wave

$$A \sin(k_x x) \sin(k_y y) \sin(k_z z)$$

k_x, k_y, k_z must obey

$$k_x = \frac{j_x \pi}{L_x} \qquad k_y = \frac{j_y \pi}{L_y} \qquad k_z = \frac{j_z \pi}{L_z}$$

where j_x, j_y, j_z are three integers. A wave in three dimensions indeed has three wavelengths: the distance it takes to repeat itself in the three directions. Frequency governs its behavior i n time, so it still only has one frequency. As you found from solving the wave equation in three dimensions, the frequency is related to the *magnitude* of the vector $\vec{k}$. The direction of $\vec{k}$ is the direction the wave is moving, and by solving the wave equation, you find that

$$\omega = c|\vec{k}| = c\sqrt{(k_x)^2 + (k_y)^2 + (k_z)^2}$$

Now call $\Delta\mathcal{N}(\vec{k})$ the number of modes which have k_x in the region $k_x, k_x + \Delta k_x$, k_y in the region $k_y, k_y + \Delta k_y$, and k_z in the region $k_z, k_z + \Delta k_z$. Then

$$\Delta\mathcal{N}(\vec{k}) = \frac{L_x L_y L_z}{\pi^3} \Delta k_x \Delta k_y \Delta k_z$$

The product $L_x L_y L_z$ is just the volume of the box. If we take the regions very small relative to the size of the box, we can write

$$d\mathcal{N}(\vec{k}) = \frac{V}{\pi^3} d^3\vec{k}$$

To know the number of modes at a given frequency, we don't care which direction the wave is going. As we saw, the frequency ω is the magnitude of $\vec{k}$, the "radial" coordinate. Thus we should integrate over all directions of $\vec{k}$. If this were a sphere, this would give us a factor 4π. However, in our construction using standing waves. k_x, k_y and k_z are positive by definition. Thus we only integrate over 1/8 of the sphere. Note that the integrand is independent of angle, and $d^3\vec{k} = k^2 dk d\Omega$. Doing the integral over angles $d\Omega$ gives

$$d\mathcal{N}(\omega) = \frac{V}{2\pi^2 c^3} w^2 dw$$

This quantity $d\mathcal{N}/dw$ is called the "density of states". If you ever take a class in statistical mechanics, you will use this formula repeatedly.

So finally we have our three ingredients: the formula for the number of photons in each level (the Bose-Einstein distribution), the energy $E = \hbar\omega$ of each photon, and the number of levels in with frequency between ω and $\omega + \Delta\omega$. Putting these together, we know that the energy in the photons with frequency between ω and $\omega + \Delta\omega$ is

$$\Delta E = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \frac{V\omega^2}{\pi^2 c^3} \Delta\omega$$

There's an extra factor of two because of the two polarizations of the photon. (You probably learned about this in E&M; the polarization of light (the direction of $\vec{E}$ or $\vec{B}$) is always perpendicular to the direction of motion, so in 3d there are two polarizations.)

This is probably the first time you've gone through a fairly elaborate derivation of an formula in physics. The reason I went through it is that it is quite important. This formula was the first theoretical explanation of an experimental result which required quantum mechanics (the fact that light comes in the form of photons is essential to get this right). It was found originally by Planck, and by fitting to the data, he found h. This is why we call h Planck's constant.

If one tries to do the computation classically, the curve just keeps increasing. The reason is that, as we saw, there are more and more modes at higher ω (note the ω^2 in the formula). Thus on general grounds, it seems that it would be more likely that the energy would be concentrated in these modes. This means that if one does the computation classically, there is no peak: the curve just keeps growing. To derive the correct formula. Planck had to assume that light came in discrete bundles, photons. The key reason is that it takes more energy to emit a photon at high ω than one at lower ω (classically, you can create arbitrarily high frequency light with any amount of energy – just reduce the amplitude). The thermodynamic factor we saw at the beginning thus supresses the higher modes exponentially, and gives the exponential falloff of the blackbody curve.

At small ω, ΔE grows as ω^2. At large ω, it decreases exponentially. There thus is a peak in between. By taking the derivative, you can find the location of the peak. You can't solve the equation for the peak value of ω analytically, but numerically you find

$$\nu_{peak} = (5.9 \times 10^{10}/K \cdot s)T$$

The surface temperature of the sun is about $6000K$. If you think of the sun as a black body (not a bad approximation), you find that ν_{peak} is right in the visible range! That is hardly a coincidence. Note that the value of the peak increases with temperature: this why e.g. metals change color as you heat them!

Another useful thing is it now tells us the total energy density of blackbodies. You just need to do the integral from $\omega = 0$ to ∞. This can be done analytically: you find the energy density is

$$U(T) = \frac{\pi^2}{15} \frac{k^4 T^4}{c^3 \hbar^3} = (7.52 \times 10^{-16} J/m^3 \cdot K^4) T^4$$

This says that a body at a temperature T has energy density growing as T^4.

So now let's think of what happens if we cut a little hole in the box. Some photons will fly out – this is called radiation. Objects at a temperature T radiate energy in the form of photons. The hotter it is, the more it radiates. Of course, if there is no energy source, it will gradually cool off. But if there is an energy source (like fusion in the sun), it keeps radiating until it runs out of fuel. At a given temperature T, we can relate the power radiated to the energy density inside. The answer is called the Stefan-Boltzmann Law, and is (in general, not just for photons) is

$$P = \frac{c}{4}U$$

The factor of c comes from the fact that the faster the stuff gets out, the more radiation there is: it should therefore be proportional to the speed of the stuff (c for photons). The factor of 4 comes because when computing how much radiation comes out of the sun, we really need the velocity perpendicular to the surface. This is $c/4$, giving the formula. You can check that the dimensions work out properly.