Modern Physics notes – Spring 2007
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Lecture 15

- Quantization of angular momentum

- Feynman, 34.2, 34.7-35.2

Quantization of angular momentum

We've now seen several examples of why quantum mechanics is called quantum mechanics. For example, the energy of the electron inside an atom is not arbitrary – it takes discrete values. The momenum of a particle inside a box likewise takes discrete values. Now we'll look at angular momentum, and find equally amazing things.

First, let's understand the classical argument for a rotating particle in a magnetic field. If you have a charged particle rotating, this is a current, which then feels a force from an applied magnetic field. The potential energy is

$$U = -\vec{\mu} \cdot \vec{B}$$

where μ is the magnetic dipole moment. Classically, μ is easy to compute, but all we need for now is the fact that

$$\vec{\mu} \propto \vec{J}$$

where $\vec{J}$ is the total angular momentum. (This remains true quantum-mechanically, but for very complicated reasons, the constant of proportionality changes).

So let's now do the Stern-Gerlach experiment, which shows what was (in the '20s) a very shocking quantum-mechanical result. What they did was put a beam of silver atoms in an inhomogeneous magnetic field. The silver atom is neutral, so the leading effect in a magnetic field comes from the dipole moment. Let's consider the beam in the $\hat{x}$-direction and $\vec{B}$ in the $\hat{z}$-direction, where B_z also depends on the value of z (this is what "inhomogeneous" means). When a potential varies in space, the particle feels a force: it wants to slide down the potential. In particular, $\vec{F} = -\vec{\nabla} U$. Here this means there is a force in the z direction:

$$F_z = -\frac{\partial U}{\partial z} = \mu \cos\theta \frac{\partial B}{\partial z}$$

where θ is the angle the dipole moment makes with z. Since the dipole moment is in the same direction as the angular momentum,

$$F_z \propto J_z$$

Thus the beam spreads out, with the atoms with the largest J_z (positive or negative) spreading out the most. Classically, you would expect that the z-component of the angular momentum could be any value up to the maximum, so the beam would just get fatter. In other words, any value of θ should appear, and so you have a continuous spread of forces.

This is not what is seen experimentally. For silver atoms, the beam just splits into two separate beams, one going up, the other going down. One can repeat this for other particles, and always finds the same thing. One can also rotate the magnets, to make sure that it isn't something funny with the source. In all cases, one obtains the same result: the beam splits into a number of pieces, evenly dispersed around the center.

What this means is that angular momentum is quantized! And by now I hope it's not surprising that you find if you plug in the numbers, that our friend $\hbar$ enters. The z component of the orbital angular momentum J_z is always quantized in units of $\hbar/2$:

$$J_z = N\hbar/2$$

where N is some integer. Notice that the units of $\hbar$ are precisely those of angular momentum. This result says that the z-component of angular momentum always comes in bundles of $\hbar/2$. This is true for *any experiment on any system* – if you ever measure J_z, you'll always get some multiple of $\hbar/2$. As with all quantum phenomena, for an everyday object you won't be able to tell: your error bars will be larger than $\hbar/2$.

Orbital angular momentum

Since we've seen several examples of quantum phenomena by now, the quantization of angular momentum may not seem so strange any more. But what I've described so far is only the tip of the iceberg of quantum weirdness.

Even in classical physics, it is common to distinguish between two kinds of angular momentum: "orbital" ($\vec{L}$) and "spin" ($\vec{S}$), so that

$$\vec{J} = \vec{L} + \vec{S}$$

The orbital angular momentum is the usual $r \times p_{CM}$ for the whole object. The "spin" is the angular momentum of the system at rest, (i.e. when its center-of-mass momentum $p_{CM} = 0$.). In terms of the earth and the sun, the earth rotating around its axis results in its spin, the rotation around the sun gives its orbital angular momentum.

The same distinction can be made in quantum mechanics. One then finds that orbital angular momentum and spin are individually quantized. Moreover, the z component of the orbital angular momentum L_z always is an *integer* number of units of $\hbar$:

$$L_z = m\hbar$$

for some integer m. This is what you showed in the homework on the Bohr atom, by making the assumption that there should be an integer number of wavelengths around the orbit. However, if you do a fully quantum-mechanical study of what's going on here, you end up with the same result.

By the way, the argument you made for the Bohr atom in the HW is called a semi-classical argument: it combined quantum and classical parts. The quantum part was the demand that the probability amplitude be periodic, while the classical part is that one still assumes that particles have well-defined orbital trajectories.

What one finds in atoms after doing things properly (i.e. no classical arguments at all) is that Bohr's argument does give the energy levels essentially correctly, and so it yields $E_n \propto 1/n^2$ in accord with the experimental results for hydrogen. However, when you measure what the angular momentum of the electron in a given level is, it misses. Here's what you find (experimentally and theoretically). For a given level n with energy $\propto 1/n^2$ is that the z component of the orbital angular momentum can take on values

$$(n-1)\hbar, (n-2)\hbar, \ldots - (n-1)\hbar.$$

i.e. $L_z = m\hbar$ where m is not the mass, but rather an integer obeying $|m| \leq n-1$. (Sorry for calling it m, but that's the convention – there are only so many letters in the alphabet.) Moreover, notice that for the lowest energy level ($n = 1$), the only possibility is that $L_z = 0$. This is something which seems to make no sense classically: in the quantum world the electron can be in "orbit" around the nucleus while having zero orbital angular momentum!

For example, for the second level $n = 2$, m can either $1, 0, -1$, so that electrons with this energy have either $L_z = \hbar, 0, -\hbar$. So the Bohr atom is completely off here: in his nth energy level, the electron has momentum $n\hbar$, which isn't even allowed here. Where the Bohr argument is correct is in the classical limit. In the atom this corresponds to n large: where the angular momentum is very large relative to $\hbar$ and the energy levels are close together. In fact, people here have recently done experiments at large n where they can "see" the electron in its classical orbit moving around the nucleus. But remember, this picture of the electron moving around the nucleus is only valid in this limit! For n small it is nothing like this: quantum effects reign supreme.

Spin

Things get weirder still when one looks at the spin, the angular momentum for a particle at rest.

If we repeated the Stern-Gerlach experiment for fundamental particles (i.e. those not made up of other known ones), the experiment measures the spin – if we put the z axis in the path, we have $\vec{r}$ is parallel to p_{CM} and so the orbital angular momentum is zero. The fact that the beam split into discrete parts was already a surprise, leading to the discovery of angular momentum quantization, described by the rules above (J_z is always a multiple of $\hbar/2$). But these rules don't explain all the strangeness: one (for atoms as well as fundamental particles) finds that for a given atom or particle not only is S_z quantized, but only takes on specific values, e.g. $S_z = -\hbar, 0, \hbar$ – no other values of S_z.

That's somewhat strange, but by looking at excited states of atoms, one can find other values of S_z (always integer multiples of $\hbar$, of course). However, let's now look at a single electron at rest. If one measures S_z, one *always* obtains either $S_z = \hbar/2$ or $S_z = -\hbar/2$. *Every electron is the same.* You never measure $S_z = 0$ for an electron, never any other integer: an electron is always spinning, and always with one of these two possibilities. Protons and neutrons have the same property. The same type of property for all fundamental particles. We say a particle has "spin s" if the allowed S_z values are

$$S_z = s\hbar, (s-1)\hbar, \ldots - s\hbar$$

where for spin s can be an integer *or* a half-integer. Thus for a given particle, all the allowed S_z are either integers, or half-integers.

Electrons, protons and neutrons have spin $s = 1/2$. A silver atom in its lowest-energy state (its "ground" state) has $s = 1/2$ as well. Photons have spin 1, although it turns out that its S_z cannot be zero: only $S_z = \hbar, -\hbar$ occur. (This comes from something you may remember about light from classical electromagnetism: it is transversely polarized, meaning that $\vec{E}$ and $\vec{B}$ are always perpendicular to $\vec{p}$.)

So this is a remarkable consequence of quantization. On the face of it, it's quite bizarre, but there is no way of escaping it. If you measure the z-component, you get one of these fixed values. Now the obvious question to ask is: what about the x component? The answers are of course the same: there's nothing special about what we chose as a z direction. The hard part is reconciling the two; and the answer we'll see is quite bizarre: the answer is that you *cannot* know both S_z and S_x at the same time!