

Lecture 16

- Spin and statistics
- Uncertainty principle for spin
- Total spin
- The periodic table
- Linear combinations

- Feynman, 5.1, 19.6

Spin and statistics

I've just explained how the spin of a particle is an *intrinsic* property. In any experiment any time any place any choice of z axis, one will measure either $S_z = \hbar/2$ or $S_z = -\hbar/2$. Other intrinsic properties of the electron include their charge and their mass, so why not spin as well? A particle's statistics is also an intrinsic property: e.g. all electrons are fermions. A very deep result (due to Pauli) relates the two intrinsic properties:

Fermions (e.g. electrons, protons and neutrons) always have half-integer S_z/\hbar . Bosons (e.g. photons) always have integer S_z/\hbar .

This result is a consequence of special relativity: the only way to make a theory of fermions mathematically consistent is if they have half-integer S_z/\hbar . This general result is quite difficult to prove – it is something even most graduate students haven't gone through.

Note that binding two half-integer-spin particles gives an integer-spin one: two fermions bind into a boson. Thus a hydrogen atom is a boson, and one will always measure $S_z = \hbar, 0$, or $-\hbar$.

Uncertainty principle for spin

Last time I explained the quantization rules for S_z . It is important to note that you can choose any particular direction and call it the z -axis: there's no z axis to the universe. So what last lecture's results mean is that once having chosen a z -axis, these values of S_z or L_z are the only ones we can measure. Note that when we say $S_z = \hbar$, we are *not* saying that the spin is pointing perfectly in the S_z direction. We are just saying that we know its z component exactly – we haven't said anything about the other components.

So what can we say about the other components? It turns out quite a lot, but since this is quantum mechanics, there is a sort of uncertainty principle. If we know S_z exactly, then we *cannot* know S_x or S_y exactly. The uncertainty principle for spin is simply

We can know at most one of the three components of spin exactly.

Say we measure $S_z = \hbar/2$ for a given electron. What I will explain precisely soon is how such a particle will be in a *linear combination* of states with $S_x = \hbar/2$ and $S_x = -\hbar/2$. If you do an experiment to measure S_x after measuring S_z , you will sometimes get $S_x = \hbar/2$ and sometimes $S_x = -\hbar/2$.

Total spin

One cannot know S_x and S_z simultaneously. However, there is another quantity you can know at the same time as S_z or any one of the three components. This is

$$\vec{S} \cdot \vec{S} = (S_x)^2 + (S_y)^2 + (S_z)^2$$

In fact, for a spin- s particle, one has

$$\vec{S} \cdot \vec{S} = s(s+1)\hbar^2$$

You can think of this as being the “total” spin, or the spin magnitude squared (note it does not have a direction). We thus know that not only does a spin- s particle have

$$S_z = s\hbar, (s-1)\hbar, \dots, -s\hbar$$

but we know its total spin as well. So this says for a given particle, you can not make it spin “faster” than its total value $s(s+1)\hbar^2$.

There's a simple argument for this result. First let's do a spin-1/2 particle. Even though we don't know if it's $S_z = +\hbar/2$ or $S_z = -\hbar/2$, for both of these possibilities, we have $S_z^2 = \hbar^2/4$.

The same goes for S_x and S_y . Thus adding the three together means that $S_x^2 + S_y^2 + S_z^2 = 3\hbar^2/4$, in agreement with the above result. For a spin-1 particle, we have two possibilities for S_z^2 : either it is \hbar^2 , or 0. If we just hand-wave and say it has an equal probability to be in any of the three states, then 2/3 of the time, $S_z^2 = \hbar^2$, while 1/3 of the time it will be $S_z^2 = 0$. The average value is thus $2\hbar^2/3$. Adding the three components together gives $\vec{S} \cdot \vec{S} = 2\hbar^2$, again in agreement with the above result for a spin-1/2 particle.

For angular momentum in general, one finds the same thing. We have

$$\begin{aligned}\vec{J} \cdot \vec{J} &= (J_x)^2 + (J_y)^2 + (J_z)^2 = j(j+1)\hbar^2 \\ J_z &= j\hbar, (j-1)\hbar, \dots, -j\hbar\end{aligned}$$

You can choose any direction you want to be the z direction: you'll always find this quantization. But once you choose the z direction, you then won't know what the values are in the x or y directions; all you'll know is this sum.

The periodic table

We now have all the ingredients we need to understand the periodic table. To label the states requires four numbers: n , l , m_l , and m_s .

The first number is n , which is any positive integer:

$$n = 1, 2, 3, \dots$$

This is the “energy level”: to first approximation all electrons with the same n have the same energy. (Things with fancy names like the “spin-orbit” coupling, and “hyperfine” splitting explain how different angular momenta have different energies, and thus shift the energy levels.)

The second number is l : this is the *total orbital angular momentum*. It is defined in the same way we defined the total spin, and the total angular momentum. Namely,

$$\vec{L} \cdot \vec{L} = l(l+1)\hbar^2$$

In general, l must be a non-negative integer. However, for atoms, for a given value of n , it turns out that one can have only certain values of l . The result (which one can show using the Schrödinger equation) is that

$$l = 0, 1, \dots, n-1.$$

This is the same label that the chemists use: they however call $l = 0$ “s”, $l = 1$ is called “p”, $l = 2$ is “d”, $l = 3$ is “f” and so on. Thus a 3d electron has $n = 3$ and $l = 2$.

The third number is m_l . This is defined so that

$$L_z = m_l \hbar$$

Thus for a given value of l , m_l can take values

$$m_l = -l, -l + 1, -l + 2, \dots, l - 1, l$$

The last number is m_s , which is the analogous quantity for spin:

$$S_z = m_s \hbar$$

Since all electrons have $s = 1/2$, we have always

$$m_s = 1/2, -1/2$$

To get the periodic table, one starts filling at $1s$ and goes from there. One interesting property is that the $4s$ electrons have less energy than the $3d$ ones. Thus potassium, which has 19 electrons, has filled $1s$, $2s$ and $2p$, $3s$, and $3p$ orbitals. This accounts for $2 + 2 + 6 + 2 + 6 = 18$ electrons. The next electron, however, does not go to to a $3d$ (i.e. $n = 3, l = 2$) level. It goes into the $4s$ ($n = 4, l = 0$) orbital. You have to go to scandium (21 electrons) before you get an electron in a $3d$ orbital.

Linear combinations

After I told you about the spin uncertainty principle, I mentioned that a state with a fixed value of S_z is a *linear combination* of states with fixed values of S_x . Let's illustrate this by using several Stern-Gerlach apparatuses. First imagine a modified one which has multiple magnets, so that the beam is first split, and then brought back together. Then you can use this apparatus to make a *filter*. By simply blocking all the paths but one, you can ensure that your beam has only one kind of S_z . Consider an atom of spin 1 and no orbital angular momentum. Then the SG apparatus splits the beam into 3 parts with $S_z = \hbar, 0, -\hbar$. If we block the bottom two, we are left with a beam made up only of atoms with $S_z = \hbar$. If we block the top and the bottom one, we are left with atoms with $S_z = 0$. To check that this interpretation is right, let's say we've blocked the bottom two, and get a beam where all particles have $S_z = \hbar$. Then we run it through a second, identical apparatus. We indeed find that the beam does not split again: the atoms remain in one beam with $S_z = \hbar$.

In terms of our notation, let $\langle b|a\rangle$ be the amplitude that a particle which goes into an apparatus in state a comes out a state b . It's important you specify which apparatus you mean; in a minute we'll tilt the apparatus, and it's important to keep track of that. Thus let's name an apparatus Z where spins feel force in the z direction. We name our three possible states $|+Z\rangle$, $|0Z\rangle$, $| - Z\rangle$. These states have $S_z = \hbar, 0, -\hbar$ respectively. (Feynman names this apparatus S , but I think the name Z is better).

Now consider a second apparatus where we block the two lower channels, so only the state $|+Z\rangle$ gets through. With this experiment, we have

$$\langle +Z|+Z\rangle = 1 \quad \text{lower two blocked}$$

Thus atoms with $S_z = \hbar$ initially will get through with probability 1. None of them will end up with $S_z = 0$ or $S_z = -\hbar$, so we have

$$\langle 0Z | +Z \rangle = 0 \quad \text{lower two blocked}$$

and

$$\langle -Z | +Z \rangle = 0 \quad \text{lower two blocked}$$

If we were to block different channels then we would get the analogous results.

Feynman details how to build an apparatus which singles out a particular component of S_z . It's important to note that the direction of the beam is not particularly important: it can be bent back to its original path by other magnets. What's important is that you can separate the different components in space, so then you can block all but one.

All this I hope is pretty obvious. The reason we've introduced this notation is that it gives a good way of understanding now what happens with the second apparatus is tilted with respect to the first. Again, coming out of this apparatus, there are three states, which we call $|+T\rangle$, $|0T\rangle$ and $|-T\rangle$. If the tilt is 90 degrees, then we're talking about the cases $S_x = \hbar, 0, -\hbar$.

So when we send the state $|+Z\rangle$ through the tilted detector, the probability for seeing each of the three tilted states depends on the tilt. Since one of these three things must happen, the probabilities must sum up to 1. In an equation,

$$|\langle -T | +Z \rangle|^2 + |\langle 0T | +Z \rangle|^2 + |\langle +T | +Z \rangle|^2 = 1$$

This is one of the strangest things about quantum mechanics: *a spin with fixed S_z contains a combination of S_x values* (and of course S_y , or any other axis you want to measure with respect to). What this means is that a beam made up purely of $S_z = \hbar$ can be thought of as a *combination* of states with $S_x = \hbar, 0$ and $-\hbar$. This is a purely quantum-mechanical effect. A state with definite S_z has undetermined S_x – all you can do is measure the various probabilities of various values of S_x .

This is somewhat akin to our double-slit experiment. There the state involved both the path through slit 1 and the path through slit 2: you couldn't say which one. Here, if you specify S_z uniquely, then you don't know which S_x value: all you can do is compute a probability.

Now you can run all sorts of experiments with multiple apparatuses. One interesting one is to first run it through Z , then T , then Z again. So say we let only $S_z = \hbar$ through the first, and $S_t = 0$ through the second, and then $S_z = \hbar$ through the third. Say N atoms get through the first. Then αN atoms get through the second, and $\alpha\beta N$ atoms make it through the third, where

$$\alpha = |\langle +Z | 0T \rangle|^2 \quad \beta = |\langle 0T | +Z \rangle|^2$$

Thus forcing it into the intermediate $|0T\rangle$ reduces the number we get out at the end. However, forcing it into the $|0T\rangle$ state means $\alpha\gamma N$ end up in the $\langle 0Z|$ state at the end, where

$$\gamma = |\langle 0T|0Z\rangle|^2$$

If we didn't force it into the $|0T\rangle$ state in the middle, we wouldn't get any of these. In fact, say we remove the blocker from the middle one (the tilted apparatus), so that everything gets through. Then there will be N atoms in the final $\langle +Z|$ state, and none in the $\langle 0Z|$ state. Thus for the latter, opening up more channels causes the net number to in that state to decrease, not increase! This is yet another example of quantum-mechanical interference.