Modern Physics notes – Spring 2007
Paul Fendley fendley@virginia.edu

**Lecture 18**

- Different components of spin

- Feynman, chapter 5

**Different components of spin**

When the beam in the $S_z = \hbar$ state went through the 90-degree-tilted Stern-Gerlach apparatus, it gets split into the three $S_x$ states. So what happens when we just send a single particle through? All we can measure is the probability that it goes into one of the three paths.

The situation here is quite similar to the particle in the box. There, we could not know its position exactly if we knew its momentum exactly (or vice versa). However, there the uncertainly principle gave us new information: the uncertainty in position told us what all the possible momentum were. Here, we can do even better. Knowing its spin component in one direction indeed leaves the others uncertain. However,

If we know $S_z$, then we know the *probability amplitudes* for $S_x$ and $S_y$.

This is what I mean when I say that $S_z = \hbar$ corresponds to a combination of $S_x$ states.

Let's understand this quantitatively for a spin-1/2 particle. Saying that it has spin-1/2 means that we can write its state as

$$|s\rangle = \alpha\,|+Z\rangle + \beta\,|-Z\rangle$$

Note that here $|\pm Z\rangle$ means $S_z = \pm\hbar/2$; I hope this isn't too confusing. This linear combination means that we don't know which of the two states it's in, but knowing it's spin 1/2 means it must be one of the two. We don't measure this combination directly: if we measure $S_z$, we get just one of the two values $\pm\hbar/2$. We get $S_z = +\hbar/2$ with probability

$$|\langle +Z|s\rangle|^2 = |\alpha|^2$$

and $S_z = -\hbar/2$ with probability

$$|\langle -Z|s\rangle|^2 = |\beta|^2$$

Since we must obtain one of these two things, the two probabilites must add to 1. Mathematically, we say that

$$\langle s|s\rangle = |\alpha|^2 + |\beta|^2 = 1$$

where we use the facts that $\langle +Z| + Z\rangle = \langle -Z| - Z\rangle = 1$, and $\langle +Z| - Z\rangle = \langle -Z| + Z\rangle = 0$. Just by knowing these two complex parameters $\alpha$ and $\beta$ lets us compute what happens when we send the beam through any combination of Stern-Gerlach apparatues.

This doesn't seem too crazy. But how we choose the $z$ axis is of course arbitrary. Thus we can repeat the last paragraph word-for-word if we subsitute $x$ for $z$: we have

$$|s\rangle = \gamma\, | + X\rangle + \delta\, | - X\rangle$$

and so on. So how do we reconcile the two statements?

Let's consider a spin-1/2 particle where we know that it has $S_z = \hbar/2$, so that $\beta = 0$. (One way of achieving this experimentally is to send it through a Stern-Gerlach apparatus with the $S_z = -\hbar/2$ path blocked.) Now send this $S_z = \hbar/2$ particle through an apparatus tilted in the $x$-direction, so that the two directions are both perpendicular to $S_z$. By symmetry, this means that half the time you measure $S_x = \hbar/2$, and half the time $S_x = -\hbar/2$. In equations, we have

$$|\langle +X| + Z\rangle|^2 = |\langle -X| + Z\rangle|^2 = \frac{1}{2}$$

so that

$$\langle +X| + Z\rangle = \langle -X| + Z\rangle = \frac{1}{\sqrt{2}}$$

Another way of writing this is to first note that

$$\langle +Z| + Z\rangle = 1$$

This is just normalization: it says that a particle with $S_z = \hbar$ has probability 1 of having $S_z = \hbar$. Likewise, we have

$$\langle +X| + X\rangle = 1 \qquad\qquad \langle -X| - X\rangle = 1$$

The non-trivial part is to then have

$$\langle +Z| = \frac{1}{\sqrt{2}}\langle +X| + \frac{1}{\sqrt{2}}\langle -X|$$

This equation summarizes the true weirdness of quantum mechanics: a particle can be in the *superposition* of states. ("Superpose" means to add on top of each other. This is not exactly that, but it's close.) This is an unavoidable consequence of the fact that a particle cannot have definite $S_x$ and definite $S_z$. If it has definite $S_z$, then it must be a *combination* of states with definite $S_x$. The same logic then says that

$$\langle -Z| = \frac{1}{\sqrt{2}}\langle +X| - \frac{1}{\sqrt{2}}\langle -X|$$

Of course, we can turn this around and write this for incoming states as well. We can also use these two to write $|\pm Z\rangle$ in terms of $|\pm X\rangle$:

$$
\begin{aligned}
\langle X| &= \frac{1}{\sqrt{2}}\langle +Z| + \frac{1}{\sqrt{2}}\langle -Z| \\
\langle -X| &= \frac{1}{\sqrt{2}}\langle +Z| - \frac{1}{\sqrt{2}}\langle -Z|
\end{aligned}
$$

and likewise for the incoming states.