Modern Physics notes – Spring 2007
Paul Fendley fendley@virginia.edu

Lecture 19

- Linear Combinations
- Bases

- Feynman, chapter 5

An example of linear combinations

Last time we saw a property of quantum mechanics which is fundamentally different from classical physics. This is the fact that a system can be in the *linear combination* of two different states. This is the way we resolve the "spin uncertainty principle": when a particle has for example a definite value $S_z = \hbar/2$, it is in a linear combination of states with $S_x = \hbar/2$ and $S_x = -\hbar/2$.

Let's go through this explicitly, for an experiment with three apparatuses, where we first run it through Z, then through a tilted apparatus tilted in the x direction X, and then through Z again. Say that in this particular experiment we let only $S_z = \hbar/2$ through the first, and $S_x = \hbar/2$ through the second, and then $S_z = -\hbar/2$ through the third. The initial state of the system is

$$|s_0\rangle = \alpha_0|+Z\rangle + \beta_0|-Z\rangle$$

If we have an SG apparatus without a wall, we don't change the overall spin: we just split the two components in space. But here the first apparatus has a wall which blocks $S_z = -\hbar/2$, so we're left with only $S_z = \hbar/2$. The state $|s_1\rangle$ after the first apparatus is therefore

$$|s_1\rangle = \alpha_0\,|+Z\rangle$$

Now we put this through the second apparatus. This splits the beam spatially into its S_t components. Even though it is entirely made up of $S_z = \hbar/2$ components, it is comprised of both components of S_x. The state *before* it hits apparatus 2 can be rewritten as

$$|s_1\rangle = \alpha_0|+Z\rangle = \frac{\alpha_0}{\sqrt{2}}\Big(|+X\rangle + |-X\rangle\Big)$$

Now it hits apparatus 2. The beam is split into its x components here, because that is the way apparatus 2 is tilted. The wall after apparatus 2 blocks all $|-X\rangle$ particles, so we're left with

$$|s_2\rangle = \frac{\alpha_0}{\sqrt{2}}|+X\rangle$$

We can now rewrite $|s_2\rangle$ before it hits apparatus 3 as

$$|s_2\rangle = \frac{\alpha_0}{2}\left(|+Z\rangle + |-Z\rangle\right)$$

Apparatus 3 blocks all the $|+Z\rangle$, leaving us only with our final state

$$|s_3\rangle = \frac{\alpha_0}{2}|-Z\rangle$$

Thus the probability it makes it all the way to the end is

$$P_{end} = |\langle -Z|s_0\rangle|^2 = \frac{|\alpha_0|^2}{4}$$

In fact, we can find the probability it makes it into through each of the apparatuses. The probability it makes it through the first apparatus is $|\alpha_0|^2$, the probability it makes it through the second is is $|\alpha_0|^2/2$.

Different bases

There's a name for the fact that we can write any state of a spin-1/2 particle can be written as a linear combination of $|+Z\rangle$ and $|-Z\rangle$. The two states $(|+Z\rangle, |-Z\rangle)$ forms one *basis* of states. Of course, we don't have to use the $|\pm Z\rangle$ states: $(|+X\rangle, |-X\rangle)$ forms another basis. Basis is actually the perfect word for this. You all know what base-10, or base 2 means. It's the same number no matter how you write it. So $|+Z\rangle$ and $(|+X\rangle + |-X\rangle)\sqrt{2}$ are just different ways of describing the same thing. The first is the most useful if you're doing an experiment to measure S_z, the second if you're doing an experiment to measure S_x.

Note that these bases make your life easier, in a way. Classically, you need three numbers to specify a spin, the S_x, S_y and S_z components. Now you need to know the coefficients of each member of some basis ($2s+1$ complex numbers). But knowing the coefficients in one basis gives us the coefficients in any basis! We've shown here how to go between the z and the x bases for a spin-1/2 particle. In Feynman you can find how to go from the z basis for a spin-1 particle to any tilt.

Why this makes life easier is that you can compute the amplitude for a given process for each basis state in any basis. Then you just need to add the amplitudes together to get the total amplitude!

To summarize these properties, label the states in a basis $|i\rangle$, and let $|\chi\rangle$ and $|\phi\rangle$ be arbitrary states. Then we have the properties:

$$
\begin{aligned}
\text{orthogonality:} \qquad & \langle i|j\rangle = \delta_{ij} \\
\text{completeness:} \qquad & |\chi\rangle = \sum_{\text{all } i} |i\rangle\langle i|\chi\rangle \\
\text{time reversal:} \qquad & \langle \phi|\chi\rangle = \langle \chi|\phi\rangle^*
\end{aligned}
$$

Orthogonality we've already discussed. Completeness means that any state can be written as the sum of basis states with some coefficients. Time reversal says that if we reverse the in and out states, the amplitude becomes the complex conjugate. This looks mysterious now, but we'll see why this is soon. Note how the first two are consistent with each other:

$$
\langle j|\chi\rangle = \sum_{\text{all } i} \langle j|i\rangle\langle i|\chi\rangle = \sum_{\text{all } i} \delta_{ij}\langle i|\chi\rangle = \langle j|\chi\rangle
$$

You should think of amplitudes like $\langle j|\chi\rangle$ (e.g. $\langle +X| + Z\rangle$) as the *overlap* of state $|j\rangle$ with state $|\chi\rangle$. It tells you that given a state $|\chi\rangle$, how much of state $|j\rangle$ is in it. In an experiment, given the state $|\chi\rangle$, you will measure the state $|j\rangle$ with probability $|\langle j|\chi\rangle|^2$.