Modern Physics notes – Spring 2007

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Lectures 20-21

- Stationary States
- Precession of a spin-1/2 particle

- Feynman, 7.1, 7.5

Stationary states

Although in our scattering experiments we discussed how probabilities depended on space, we haven't said much about time. On a homework, you did derive that the uncertainty principle for energy and time is

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Thus the more certain you are about energy, the less you know about what at which time some event happens.

The simplest behavior is a "stationary state". This is a state where the probabilities are unchanging in time. Because of the uncertainty principle, a state unchanging in time means effectively that $\Delta t = \infty$: it spreads out over all time. (This may seem a little weird, but if you think of a system which is the same over all space, then $\Delta x = \infty$.) When $\Delta t = \infty$, ΔE is arbitrarily small. This means a *stationary state has definite energy.* This is why we can talk about energy levels. A stationary state has *probabilities* unchanging in time. This does not necessarily mean that the *amplitudes* are independent of time, since the probabilities are given by their magnitude squared. Just like a free particle of definite momentum $\vec{p}$ has amplitude proportional to $e^{i\vec{p}\cdot\vec{r}/\hbar}$, the amplitude of a state of definite energy E depends on time as

$$e^{-iEt/\hbar}$$

We'll see from relativity that this is not a coincidence, but must be this way.

We'll get "interference in time" just like we got interference in space. In space, interference could occur when a particle could follow two different paths. Interference in time can occur when we have a state that is a mixture of different energies. Remember, this is quantum mechanics – a particle can be in the linear combination of different states.

The precession of a spin-1/2 particle

That's all pretty abstract. To see how to apply this, let's look at a spin 1/2 particle (electron, proton, neutron) in a magnetic field. This will also let us see an example of what I called earlier "interference in time". Moreover, the physics here applies to many two-state systems like masers and lasers.

First, let's discuss this classically. Say we start out with a stationary charged particle spinning with angular momentum $\vec{J}$ in a magnetic field $\vec{B}$ direction. As we discussed before, its magnetic moment $\vec{\mu}$ is in the same direction as the angular momentum, and using $\vec{F} = q\vec{v} \times \vec{B}$, you can work out that the energy U and torque $\vec{\tau}$ are

$$\begin{aligned} U &= -\vec{\mu} \cdot \vec{B} \\ \vec{\tau} &= \vec{\mu} \times \vec{B} \end{aligned}$$

You can look in Feynman volume 2 for a proof of these, but its pretty easy to see what's going on by drawing the picture. The meaning of the first term is simple: the magnetic field wants to align the spin with it. There is no net force on a stationary particle spinning in a uniform magnetic field, only a torque (remember the Stern-Gerlach experiment requires a non-uniform magnetic field).

First, say the magnetic field is pointing in the same direction as the spin (i.e. the axis of rotation). Then there is no torque. By looking at the energy, we see that this is a mininum of energy is stable. If the spin is in the exact opposite direction as the magnetic field, then there is also no torque. This is a maximum of energy, and so it is unstable (like a particle on top of a hill).

However, things are a little subtler with a particle spinning perpendicular to the magnetic field. For example, place a particle with spin (axis of rotation) in the x direction in a uniform magnetic field in the z direction. One end of the particle feels a force in the $+\hat{x}$ direction, while the opposite end feels a force in the $-\hat{x}$ direction. This causes a torque, giving the particle angular momentum around the y axis. We thus say the spin *precesess*. This is just like what happens with a gyroscope. Keep applying the torque, and all the angular momentum will end up in the y direction. Once the angular momentum is in the y direction, the torque from B_z wants to make the angular momentum in the x direction, so the spin's axis of rotation goes around. Note that as long as the axis of rotation points anywhere in the x-y plane, the energy remains zero, because $\vec{\mu} \perp \vec{B}$.

So now let's see what happens quantum-mechanically. If the $|+Z\rangle$ state has magnetic moment μ_z, then the $|-Z\rangle$ state has magnetic moment $-\mu_z$. Basically, the magnetic field wants to align the spin with it. The spin-up state has definite energy $-\mu_z B_z$, while the spin-down state has definite energy $\mu_z B_z$. Now recall that a state of definite energy evolves in time as $e^{-iEt/\hbar}$. Thus for a $\vec{B}$-field pointing in the z-direction, the states $|\pm Z\rangle$ evolve in time as $e^{\mp i\mu_z Bt/\hbar}$. They don't change in time, because $|e^{\mp iBt/\hbar}| = 1$. This is exactly the same as happens classically.

So now let's consider what happens when we have a state which starts out with $S_x = \hbar/2$, but is placed in a magnetic field in the z direction. To solve this problem, it is convenient to use not the basis $(|+X\rangle,\ |-X\rangle)$, but instead the basis $(|+Z\rangle,\ |-Z\rangle)$. Because this basis is complete, we can write an *arbitrary* state of a spin-1/2 particle as

$$|s(t)\rangle = C_1(t)|+Z\rangle \ + \ C_2(t)|-Z\rangle$$

where $C_1(t)$ and $C_2(t)$ are functions we will compute for this case. The state will of course depend on time because we are putting a torque on it and causing it to precess. Of course, we can rewrite this in the x basis (and will do so below). The reason we want to work in the S_z basis is that we know how these states evolve in time in a magnetic field in the z direction.

Let's define the time $t = 0$ to be the time where the spin has definite $S_x = \hbar/2$. In the S_z-basis, we showed earlier that we can write this as

$$|+X\rangle = \frac{1}{\sqrt{2}}|+Z\rangle \ + \ \frac{1}{\sqrt{2}}|-Z\rangle$$

This means that at $t = 0$, we have $C_1(0) = 1/\sqrt{2}$, and $C_2(0) = 1/\sqrt{2}$. But because we know the time evolution for these states goes as $e^{\mp i\mu_z Bt/\hbar}$, we have

$$C_1(t) = \frac{1}{\sqrt{2}}e^{i\mu_z Bt/\hbar} \qquad\qquad C_2(t) = \frac{1}{\sqrt{2}}e^{-i\mu_z Bt/\hbar}$$

This determines our state for all time!

First, let's compute the amplitude as a function of time the particle is in the $S_x = \hbar/2$ and $S_x = \hbar/2$ states. To do this, it is now convenient to go back to the x-basis. We have

$$\begin{aligned} C_1(t)|+Z\rangle \ + \ C_2(t)|-Z\rangle \ &= \ \frac{C_1(t)}{\sqrt{2}}\Big(|+X\rangle \ + \ |-X\rangle\Big) \ + \ \frac{C_2(t)}{\sqrt{2}}\Big(|+X\rangle \ - \ |-X\rangle\Big) \\ &= \ \frac{C_1(t)+C_2(t)}{\sqrt{2}}|+X\rangle \ + \ \frac{C_1(t)-C_2(t)}{\sqrt{2}}|-X\rangle \\ &= \ \cos(\mu_z Bt/\hbar)|+X\rangle \ + \ i\sin(\mu_z Bt/\hbar)|-X\rangle \end{aligned}$$

This is quantum-mechanical precession! At $t = 0$, the particle is indeed in the x direction (this was our initial condition). The amplitude repeats itself with period $2\pi/(\mu_z B)$. But notice that at times $(n + 1/2)\pi/(\mu_z B)$, it is entirely in the $-\hat{x}$ direction. This means that the probability the particle is in the $S_x = \hbar/2$ state is

$$P_{+x} = |\langle +X|s(t)\rangle|^2 = \cos^2(\mu_z Bt/\hbar)$$

and the probability to be in the $S_x = -\hbar/2$ state is

$$P_{-x} = |\langle -X|s(t)\rangle|^2 = \sin^2(\mu_z Bt/\hbar)$$

The probabilities add to 1 as they must: if you measure S_x, you find either $\hbar/2$ or $-\hbar/2$, but nothing else.

Let me clarify what we have done. An arbitrary state of a spin-1/2 particle can be expressed as the combination of the "spin-up" $S_z = \hbar/2$ and the "spin-down" $S_z = -\hbar/2$ states. The key trick in doing this problem is to note that when the purely $S_z = \hbar/2$ state is a stationary state when the magnetic field is in the z direction. The $S_z = -\hbar/2$ state is a stationary state. However, *their linear combination is not.* The reason is that the $S_z = \pm\hbar/2$ stationary states have different energies, and so they evolve differently in time. Thus any combination of them does not remain unchanging in time. This is the precise meaning of "interference in time".