Modern Physics notes – Spring 2007

Paul Fendley fendley@virginia.edu

Lecture 30

- Lorentz transformations

- Born, chapter VI
- Fowler, "Special Relativity", "Time dilation and length contraction"

Lorentz transformations

The loss of absolute simultaneity is just one of the dramatic consequences of the speed of light being the same in any inertial frame. We will see how people in different frames will measure different lengths for the same extended object, and different time intervals between two events. These two effects are called *length contraction* and *time dilation*, and we will derive them today.

The general way of relating a measurement in one inertial frame to that in another is called a Lorentz transformation. Precisely, a Lorentz transformation tells you how some event at a point (x, t) in spacetime in our frame is described as a point (x', t') in an intertial framoving with respect to us. Born gives two derivations. The first uses the graph I defined at the end of the last lecture, while the second is algebraic. Here I'll follow the second, but it's worth looking at the first as well.

The "world-line" of a particle is the plot of its trajectory in space and time. The simplest case is a particle at rest in our frame. This gives a vertical world line: a line parallel to the ct axis. Now consider the world-line of a particle at rest in S' at a point C'. In S, this world line is described as $x - vt = C$ for some constant C. Thus the line $(x' = C', t')$ in the moving frame is the same line as $(x = C + vt, t)$ in the stationary frame. Thus define

$$\gamma \equiv \frac{C'}{C} = \frac{x'}{x - vt}$$

This means that

$$x' = \gamma(x - vt)$$

Note we have defined things so that the origins of both frames coincide at $t = 0$ and $t' = 0$.

Now consider the same situation, from the point of view of the other frame. They think we're moving at velocity $-v$ with respect to them. They would define γ from their point-of-view, where

v is replaced by $-v$, and the roles of the frames are reversed. Thus here

$$\gamma = \frac{x}{x' + vt}$$

$$x = \gamma(x' + vt')$$

We now apply the principle of relativity, which says that there's nothing particularly special about our frame or any given inertial frame. This means that while of course γ can depend on v (since this is the relative velocity between the two frames), it should only depend on the *magnitude* of v, not its direction. If it did depend on direction, this would amount to having a preferred direction, which would violate of our first principle. Thus this is the *same* γ that appears above, because γ only depends on $|v|$ (or $|\vec{v}|$ in higher dimensions).

We can use the two equations to solve for t' in terms of the other variables to get

$$t' = \frac{x - \gamma x'}{v\gamma} = \frac{1 - \gamma^2}{\gamma v} x + \gamma t$$

Likewise, one can solve for x'. As long as $\gamma \neq 1$, we have $t \neq t'$. A key novelty of relativity is that t' is *not* the same as t: the time between two events is measured differently in the two frames.

These two equations relate any event (x, t) in one frame to its description (x', t') in another frame. To make these equations useful, we need to find γ and show that it indeed is not 1. To do this we need to use the second of our principles. So emit a light burst at $x = 0$ at time $t = 0$. We have set up our frames so that this will be $x' = 0, t' = 0$ in the other frame. The second principle says that observers in any frame will measure the same speed c. Thus in our frame at some later time t, we'll measure the position of the light to be $x = ct$ in our frame (the second event). An observer on the train can watch us do this measurement, but they'll measure the second event to be at position $x' = ct'$ at time t'. Plugging these into our above equations, we can find γ.

$$\begin{aligned} ct' &= \gamma(c - v)t \\ ct &= \gamma(c + v)t' \end{aligned}$$

Multiplying the two makes the factors of t and t' cancel, giving

$$c^2 = \gamma^2(c^2 - v^2)$$

or

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Note the factor γ always obeys $\gamma \geq 1$, and it goes to infinity as $v \to c$. This is our first sign that nothing can go faster than light – if $v > c$, γ is imaginary!

We can use the relation for γ to simplify the above relations between (x, t) and (x', t'), giving the *Lorentz transformations*

$$x' = \gamma(x - vt), \qquad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

These formula apply to a *single event*. They relate the spacetime point of this event (x, t) in the frame S to the spacetime point (x', t') in a frame S' moving at a velocity v with respect to S. As a simple check, note that if $v = 0$ (the frames are not moving with respect to each other), then $\gamma = 1$ so that $x' = x$ and $t' = t$.

We've just been discussing one dimension. Extending it to three dimensions is pretty simple: just take the motion in the x direction. Then we simply have $y' = y$ and $z' = z$. By convention we'll always take the relative motion between two frames to be in the x direction.

We can write formulas for (x, t) in terms of (x', t') instead. One can just solve for the two in these two equations, but it's easier to get the answer by just noting that the frame S is moving with respect to S' with velocity $-v$. Thus we just interchange primed and unprimed and change $v \to -v$ to get

$$x = \gamma(x' + vt'), \qquad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

Note that γ only depends on v^2, so it is the same here and above.

The Lorentz transformation relates specific events. By considering two events, we can use Lorentz transformations to relate lengths and time intervals in the two frames S and S'. So let's consider a rod of length L at rest. We want to find what length L_{move} the rod will have if we measure it while it's moving. The reason $L \neq L_{move}$ is the loss of simultaneity. By "measure" a rod, we mean find the locations of the two ends at the *same* time, i.e. simultaneously. Because what's simultaneous in one frame is not in another, this means that the rod is measured differently in the two frames. In other words, measuring of the rod requires two events, and the two events are not the same in different frames.

What happens is that $L_{move} < L_{rest}$: someone who measures the rod while it is moving will measure a shorter distance then the person who measures it at rest. This is called *length contraction*. Say the rod is at rest in the frame S'. Place one end of the rod at the origin $x' = 0$, so the other end is at $x' = L'$, where L' is the length of the rod at rest. We then have

$$0 = \gamma(x_1 - vt), \qquad L' = \gamma(x_2 - vt)$$

where x_1, x_2 are the positions of the ends of the rod in our frame at a time t. (It is moving, so they are not constant). We then measure the rod length in our frame. The catch is that we measure both ends of the rod at the same time in our frame. Say we measure it at $t = 0$. Then one end is at $x_1 = 0$, the other is at $x_2 = L'/\gamma$. The length L we measure is

$$L = x_2 - x_1 = L'/\gamma$$

Since $\gamma \geq 1$, we see the length as smaller.

Another effect is called *time dilation.* For example, let this be the time interval between the creation of a muon in a collision in the upper atmosphere, and when it decays (into an electron and two neutrinos). The muon is moving toward us at some velocity v, so it is the rest frame: T_{rest} is the lifetime it measures, while T_{move} is the time we would measure, since we of course see these two events at different places. Make the creation happen at the origin. The decay happens at some time t' at $x' = 0$. This means that x for the second event is given by $x = vt$. Plugging this into the Lorentz transformation for time gives

$$t' = \gamma \left(t - \frac{v}{c^2} vt \right) = t \frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} = \frac{t}{\gamma}$$

Since $T_{rest} = t' - 0$ and $T_{move} = t - 0$, we have

$$T_{move} = T_{rest}\gamma$$

Since $\gamma \geq 1$, it looks to us that the muon lives a longer time. Since in principle γ can be arbitarily large, this is how you can "time travel" into the future!