Modern Physics notes – Spring 2007

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Lecture 31

- addition of velocities
- Lorentz invariants

- Born, chapter VI

Addition of velocities

I showed before that nothing can go faster than the speed of light. But this then opens up a question. Say a particle decays, and emits two particles moving in opposite directions at very large speeds, say 3/4 the speed of light. There's nothing inconsistent with this, and it happens all the time. But now let's ask question: what one would one particle measure the other's velocity to be? Naively, it seems to be 3/2 the speed of light. This violates the speed limit of c (the frame of one the particles is a perfectly good inertial frame, so what it sees is subject to the same laws of physics as we are).

Thus let's go back to the train moving to the right at speed v, and say someone on the train shoots a bullet in the x direction a velocity u'_x with respect to the train. So what do we measure the bullet's velocity u to be? We've just seen that we can't have $u_x = v + u'_x$, or this will violate the speed limit of c. The Lorentz transformations we derived let us find the correct answer. For simplicity, let the bullet be emitted at $x = 0, t = 0$, which coincides with $x' = 0, t' = 0$ by definition. Then the location of the bullet depends on time in the two frames as

$$x = u_x t \qquad \text{and} \qquad x' = u'_x t'$$

So let's put this into our Lorentz transformations.

$$u_x = \frac{x}{t} = \frac{\gamma(x' + vt')}{\gamma(t' + vx'/c^2)} = \frac{(u'_x + v)t'}{(1 + vu'_x/c^2)t'} = \frac{u'_x + v}{1 + vu'_x/c^2}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ as usual. So note that as we take $u'_x \to c$, u just approaches c as well. For $u'_x = c$ (i.e. it emits a light beam instead of a bullet), we have

$$u_x = \frac{c + v}{1 + vc/c^2} = c$$

as it must: everybody measure the speed of light. This is also a proof that the speed can never exceed c. Thus everything is consistent: the factor γ never becomes imaginary.

The above formula applies to the case where the two velocities you're adding are in the same direction (the bullet was fired in the direction of the train). To get the general formulas, say the bullet wasn't fired in the x direction, but has a component of velocity in the y direction u'_y. The y coordinate is therefore given by $y' = u'_y t'$ and $y = u_y t$ in our frame. We said before that $y' = y$, but of course $t' \neq t$. The y velocity is therefore given by

$$u_y = \frac{y}{t} = \frac{y'}{\gamma(t' + vx'/c^2)} = \frac{u'_y}{\gamma(1 + vu'_x/c^2)}$$

Again, it cannot get larger than c. A check on this fairly ugly formula is that $u_y = u'_y$ when $v = 0$.

Lorentz invariants

One interesting thing to note is that even though the time and the distance between two events is measured to be different in two different frames, there is a quantity which is the same in both. Say one event happens at $(\vec{x}_1, ct_1)$ and the second at $(\vec{x}_2, ct_2)$ We then say they're a distance $|\vec{x}_2 - \vec{x}_1|$ apart, and separated by the time interval $t_2 - t_1$. In another frame moving at velocity v, they'll of course measure $|\vec{x}'_2 - \vec{x}'_1|$ and $t'_2 - t'_1$ respectively. But consider the combination

$$\Delta = c^2(t_2 - t_1)^2 - |\vec{x}_1 - \vec{x}_2|^2$$

(This is the opposite of the quantity F defined in Born.) By plugging in the Lorentz transformations, you can see that

$$\Delta = c^2(t'_2 - t'_1)^2 - |\vec{x'_1} - \vec{x'_2}|^2$$

with the *same* Δ. This is what we call a Lorentz invariant – every frame will get the same answer for this quantity.

Lorentz invariants like Δ are useful. Δ, for example, is useful for classifying pairs of events. In particular, it gives a way of telling: can one event influence the other? First of all, consider emitting a light burst at time t_1 at some point x_1 and measuring it at time t_2 at some location x_2. Then by the constancy of the speed of light, we have $x_2 - x_1 = c(t_2 - t_1)$ in this frame. Since in another frame, an observer sees the light moving at the same speed c, we would have for these two events $x'_2 - x'_1 = c(t'_2 - t'_1)$. Both frames have $\Delta = 0$. This allows us to distinguish between three types of event pairs. If $\Delta = 0$, then we say the events are *light-like separated.* The only way to get from one event to the other is to travel at the speed of light – anything else will travel slower, and so won't make it.

If $\Delta > 0$, then the events are *time-like separated.* Here it is possible to make it from one event to another; one event can cause the other. In fact, for $\Delta > 0$ we can find a frame where they happen at the same place in space: just find a frame $\widetilde{S}$ so that in this frame $|\widetilde{x}_2 - \widetilde{x}_1|^2 = \sqrt{\Delta}$ When $\Delta > 0$, this is is always possible. However, in this case you can never find a frame where the events are simultaneous ($t_2 = t_1$), because this would then require $\Delta < 0$. This is the reason for the name – the two events are always separated in time.

For $\Delta < 0$, the events are *space-like separated*. We can find a frame where they are simultaneous, but we can't find one where they're at the same spatial point. Space-like separated events can *not* be caused by one another: even light cannot travel fast enough to get from one another. This was what was so weird about the EPR "paradox" we discussed. The two measurements happen at the same time but a large distance away. There's no way a light signal can get from one to the other. They way this effect is consistent with relativity is only because no information can get from one to another. (To fully understand how this can be, you need to combine relativity and quantum mechanics, which is the subject of quantum field theory.)