Modern Physics notes – Spring 2007
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Lecture 32

- $E = mc^2$ without relativity
- Born, chapter VI

$E = mc^2$ without relativity

One of the most famous formulas in physics is $E = mc^2$, first derived by Einstein. It says, quite remarkably, that mass is a form of energy. Energy is still conserved, but now you can think of mass as a form of potential energy. Thus if you can make it so the system loses mass, then this energy can be converted to other forms. This is the principal behind atomic power: if you split an atom into two smaller atoms whose combined mass is less than the original, you obtain (lots of) energy. The reason you obtain lots of energy is the fact that in standard units, the factor c^2 is enormous.

There's a full-blown formula for the energy and momentum of a particle in a relativistic theory. We'll get to that soon. But $E = mc^2$ can be derived without recourse of all this relativistic formalism. The argument is from Einstein, and it's in Born. I'll review it here, in a slightly different. I'll use the language of quantum mechanics, but that's totally unnecessary – it's a purely classical argument.

Consider a train car of mass M and length L, completely symmetric around its middle, with one exception. One one side there's an atom in some excited state, on the other side the atom is in its ground state. The atoms differ by some energy ΔE. The atom falls from the excited state to the ground state, emitting a photon of energy ΔE, which moves over to the other side and excites the opposite atom from the ground state into the excited state. Because light has momentum as well as energy, this photon has momentum $p = \Delta E/c$. Thus when the photon is emitted, the train car must *recoil* with momentum $-p$. This gives the train a velocity $v = -p/M$. Now when the photon is absorbed on the other side, the train re-absorbs this momentum and thus stops. But while light is moving from one end to the other, the train is moving with velocity v. It takes a time $t = L/c$ to get from one end to the other, so the train moves a net distance

$$d = vt = -\frac{p}{M}\frac{L}{c} = -\frac{(\Delta E)L}{Mc^2}$$

So this seems to be no big deal. But recall that in mechanics, a basic principle is that the only way to make an object accelerate is to apply an external force. Here there are no external forces, so how does the object move?

The answer of course is that what doesn't move is the center of mass. The object can change shape without external forces, as long as its center of mass doesn't move. Thus when the photon was emitted and then reabsorbed at the other end, the train may move, but the center of mass must stay the same. The only way this is possible is if the atom in an excited state weighs more than it does in its ground state. Let's say that the atom in its ground state has some mass, but that in its excited state its mass is greater by some amount Δm. If mass weren't a form of energy, than of course $\Delta m = 0$, but we'll see here that's not the case. So see how we can keep the center of mass in the same place. Remember the formula for center of mass of two objects is

$$x_{cm} = \frac{1}{m_1 + m_2}(m_1 x_1 + m_2 x_2)$$

The two objects in this problem are the train itself, of mass M and the excited atom, of "mass" Δm. We include the mass of the atoms themselves in the train mass M, so that we have $m_1 = M$, and $m_2 = \Delta m$. The train moving over a distance d shifts $x_1 \to x_1 + d$, while the change in the location of the excited atom shifts $x_2 \to x_2 + L$. The fact that x_{cm} must be unchanged in this process means that

$$0 = \frac{1}{M + \Delta m}(Md + \Delta m L)$$

Thus $dM + L\Delta m = 0$, or the unknown "mass"

$$\Delta m = -\frac{Md}{L} = \frac{(\Delta E)L}{Mc^2}ML = \frac{\Delta E}{c^2}$$

Thus an atom excited by an energy ΔE higher weighs more! The extra weight is exactly $\Delta E/c^2$ – note that this is independent of what kind of atom we have, or for that matter, is independent of everything else other than the speed of light. Relativity shows how this must be completely general: if the mass of something changes by Δm, then its total energy changes by ΔE, where

$$\Delta E = (\Delta m)c^2$$

That's it! Mass is a form of potential energy.