Modern Physics notes – Spring 2007

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Lecture 33

- Relativistic momentum and energy

- Born, chapter VI.6-9

Relativistic momentum

Since we've found that velocities do not add in the Newtonian way, this means that our formulas for momentum and energy are called into question. However, we can use our velocity-addition formula to find their relativistic versions. Energy we know is going to be tricky, because we have to get $E = mc^2$ somehow, so let's look at momentum first. Momentum is a vector, and should depend on no properties of the system other than velocity and mass. Thus the relativistically-valid momentum must be of the form

$$\vec{p} = A(|\vec{v}|, m)\vec{v}$$

where we need to determine the function A, which depends only on m and the magnitude of $\vec{v}$. From non-relativistic physics, we know that for $\vec{v}$ small, $A \approx m$, so $A(0, m) = m$.

To find the function A, consider a situation where we have two identical balls of mass m colliding. They stick together, and move off. First consider the frame where one ball is resting and the other moving at a velocity v in the x direction. Then the initial momentum of the whole system is

$$p_x = A(v_x, m)v_x$$

The final blob moves off at some velocity u_x, so we have the final momentum (which must be the same as the initial)

$$p_x = A(u_x, m_{blob})u_x$$

If this were non-relativistic, we would just have $u_x = v_x/2$ and $m_{blob} = 2m$, but later we'll show neither is true in the relativistic theory. We can find u_x by using the addition-of-velocities formula we derived last time. Consider a frame S' which is moving to the right at velocity v_x. Now the ball on the left is motionless and the ball on the right is moving at a velocity $-v_x$. By symmetry, we must have the final velocity $-u_x$. However, our addition-of-velocities formula relates u_x to v_x. In frame S, the blob moves at velocity u_x. In frame S', moving at velocity v_x with respect to S, we have velocity $-u_x$. The addition of velocities formula says that

$$u_x = \frac{-u_x + v_x}{1 - u_x v_x/c^2}$$

Solving for v gives

$$v_x = \frac{2u_x}{1 + u_x^2/c^2}$$

Solving for u_x instead gives

$$u_x = \frac{c^2}{v_x}\left(1 - \sqrt{1 - v_x^2/c^2}\right)$$

Note that if c is very large, $v_x \approx 2u_x$, as we expect non-relativistically.

So to get the function A, we need to do a clever trick. This is now to change frames to one which is moving by $-v_y$ in the y direction. Both the initial particles and the blob are moving in the y direction with a velocity v_y. (Remember, we're doing the same experiment, just looking in a different frame.) The y momentum initially is then

$$p_y = A(\sqrt{v_x^2 + v_y^2}, m)v_y + A(v_y, m)v_y$$

and finally

$$p_y = A(\sqrt{u_x^2 + v_y^2}, m_{blob})v_y$$

Conservation of y momentum then gives

$$A(\sqrt{v_x^2 + v_y^2}, m) + A(v_y, m) = A(\sqrt{u_x^2 + v_y^2}, m_{blob})$$

This must be true for any choice of v_x, v_y and u_x. For example, for $v_y = 0$ we have

$$A(v_x, m) + A(0, m) = A(u_x, m_{blob})$$

So A (whatever it is) is also conserved!

Let's figure out what the function A is, so we can figure out what relativistic momentum is, and what our new conservation law is. Our original momentum conservation equation reads

$$A(v_x, m)v_x = A(u_x, m_{blob})u_x$$

We just worked hard to derive that $A(u_x, m_{blob}) = A(v_x, m) + A(0, m)$, so let's use this first to get rid of $A(u_x, m_{blob})$ and solve for $A(v_x, m)$. We have

$$A(v_x, m) = A(0, m)\frac{u_x}{v_x - u_x}$$

To simplify this a bit, remember that for small velocities (the non-relativistic limit), we need to get good old $p = mv$. This means that $A(0, m) = m$, so

$$A(v_x, m) = m\frac{u_x}{v_x - u_x}$$

Now finally, we just need to use our relation of the final velocity of the blob u_x to the initial velocity v_x. Plugging this in and doing a little algebra (which you can find in Born), we find a simple answer:

$$A(v, m) = m\gamma$$

where γ is our old friend $\gamma = 1/\sqrt{1 - v^2/c^2}$.

This means that the relativistic equation for momentum is

$$\vec{p} = \gamma m \vec{v}$$

This is conserved in the absence of external forces. For $v \ll c$, $\gamma \approx 1$, and this reduces to the formula you're used to. It's important to remember that everything in relativity reduces to the Newton we know and love in the limit that all velocities are much smaller than c. This often provides an excellent check on whatever computation you're doing.

Sometimes people (including Born) call γm the "effective mass" m_{eff} of a moving particle, and m the "rest mass", that we can still use $p = m_{\text{eff}} v$ even in relativity. Sometimes they even call "effective mass" just plain mass, which can lead to confusion. But this also can make things clearer. For example, it gives an intuitive way of understanding why you can't accelerate a particle to faster than the speed of light: as $v \to c$, the moving "mass" gets larger and larger, because $\gamma \to \infty$. So no force is strong enough to accelerate the particle more. But to avoid getting confused it's always good to call $m\gamma$ the effective mass.

Relativistic Energy

Let's figure out what conservation of A means. We derived

$$A(v_x, m) + A(0, m) = A(u_x, m_{blob})$$

and now we know that $A(v, m) = m\gamma$. It's always a good idea to understand things in simple limits, so let's look at what happens when $v \ll c$. Taylor's formula says that for z small, we have

$$\frac{1}{\sqrt{1-z}} = 1 + \frac{1}{2} z + \ldots$$

Thus for v/c small we have

$$\gamma = \left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right)$$

So we have

$$A = m\gamma = m\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right)$$

So let's define a quantity

$$E(v_x, m) \equiv A(v_x c^2)$$

Since we're merely multiplying A by a constant, this funny new quantity is conserved just like A is. Expanding out E for small v, we have

$$E(v_x, m) = mc^2 + \frac{1}{2} m v^2 + \ldots$$

Look familiar?

Thus conservation of A is just conservation of energy. The new ingredient is that there is a piece which is independent of velocity, the "rest energy"

$$E_{rest} = mc^2$$

This is just Einstein's formula – mass is a form of energy! The next piece is just non-relativistic kinetic energy. But now we now have a simple relativistically-valid formula for the full energy,

$$E = \gamma mc^2 = E_{rest} + E_{kinetic}$$

where $E_{kinetic} = (\gamma - 1)mc^2$. Expanding out $\gamma - 1$ in powers of v/c gives the non-relativistic kinetic energy $mv^2/2$ we're used to. So relativity lets us combine the conservation of mass and kinetic energy beautifully into one simple formula.

There are several questions to ask. First of all, you'll remember that in an inelastic collision like the one we discussed, the kinetic energy $1/2mv^2$ isn't conserved. So how do we get conservation of energy here? The second is that we still need to understand if m_{blob} is equal to $2m$ or not. Remember the train-car argument for $E = mc^2$: exciting an atom changes its mass.

The two questions basically have the same answer. Since momentum is conserved even in inelastic collisions, let's focus on that. To summarize the long argument, we saw that conservation of momentum in the x direction means that

$$A(v, m)v = A(u, m_{blob})u$$

We used the trick with the boost to relate u to v via

$$v = \frac{2u}{1 + (u)^2/c^2}$$

We used conservation of y momentum to get the conservation of energy

$$A(v, m) + A(0, m) = A(u, m_{blob})$$

Putting all this together gave us $A(v, m) = m\gamma_v$ and so we interpret

$$Ac^2 = \gamma mc^2 = mc^2 + \frac{1}{2}mv^2 + \ldots$$

as the total energy.

So now let's see what this gives for m_{blob}. Doing a little algebra gives

$$A(u, m_{blob}) \quad = \quad A(0, m) + A(v, m)$$

$$
\begin{aligned}
&= m\left(1+\frac{1}{\sqrt{1-v^2/c^2}}\right) \\
&= m\left(1+\frac{(1+u^2/c^2)}{\sqrt{(1+u^2/c^2)^2-4u^2/c^2}}\right) \\
&= m\left(1+\frac{1+u^2/c^2}{1-u^2/c^2}\right) \\
&= \frac{2m}{1-u^2/c^2} \\
&= 2m(\gamma_u)^2
\end{aligned}
$$

So what does this mean? Recall we have $A(v,m)=m\gamma_v$. Thus if we want $A(u,m_{blob})=m_{blob}\gamma_u$, we must identify

$$m_{blob}=2m\gamma_u$$

In the non-relativistic limit, we obtain the usual $m_{blob}\approx 2m$, but for large enough u, this does not hold. In fact, the blob *grows* in mass (because $\gamma>1$)!

How can this be? Recall the other question: in an inelastic collision, the final kinetic energy is smaller than the initial. One always explains this by saying that this goes into heat. Well, we saw in our $E=mc^2$ argument that mass is energy, and energy is mass. Thus if we heat an object, it weighs more! This is why $m_{blob}>2m$: the extra mass is the heat in the object. Eventually, this heat will presumably radiate away, and the mass will fall to $2m$, or possibly even lower (since when you stick them together, you rearrange some chemical bonds, the "final" mass may be larger or smaller than $2m$.

To summarize, a particle of mass m and velocity $\vec{v}$ has momentum $\vec{p}=\gamma m\vec{v}$, rest energy $E=mc^2$, and kinetic energy $(\gamma-1)mc^2$.

We can put find a simple relation between the relativistic formulas for momentum and energy. Note that

$$|\vec{p}|^2=p_x^2+p_y^2+p_z^2=\gamma^2m^2(v_x^2+v_y^2+v_z^2)=\gamma^2mv^2$$

This means that

$$E^2-c^2|\vec{p}|^2=m^2\gamma^2c^2(c^2-v^2)=m^2c^4$$

This says that if we define E and p in any frame by the formulas $E=\gamma m$ and $\vec{p}=\gamma m\vec{v}$, when we take the above combination, we always get m^2c^4. This means rest mass is a Lorentz invariant: if we take the combination $(E')^2-c^2|\vec{p'}|^2$ in any frame, we always get m^2c^4.

Note the similarity between this and our earlier Lorentz invariant $c^2(t_1-t_2)^2-|x_1\vec{-}x_2|^2$. This leads to an very useful way of looking at things called the *four-vector*. Just like we defined a spacetime point by $(\vec{x},ct)$, we define the "energy-momentum" four-vector as $(\vec{p},E/c)$. All four-vectors transform in the same way under Lorentz transformations. In other words, we write down formulas for the momentum and energy in different frames by substituting $\vec{x} \to \vec{p}$ and $ct \to E/c$ in the Lorentz transformation formulas. Another four-vector is $(c\rho, \vec{j})$, where ρ is the charge density and $\vec{j}$ the current density. Another is $(\phi, \vec{A})$, where ϕ and $\vec{A}$ are the electrostatic and vector potentials of electromagnetism.

Thus we have a simple formula relating the energy to the momentum:

$$E = \sqrt{p^2c^2 + m^2c^4}$$

The great thing about the formula is that it applies for all particles. For photons $m = 0$, so we recover $E = pc$. For massive particles with $v \ll c$, expanding this for $p \ll mc$ gives

$$E = mc^2 + \frac{p^2}{2m} + \dots$$

Look familiar?

This concept of rest mass has profound consequences for civilization. The point is that if we can split or combine nuclei in such a manner that the final state weights less than the final state, there is an amount of energy released

$$E_{released} = (m_{initial} - m_{final})c^2$$

Since c^2 is so large, a very small difference in mass can result in a lot of energy. You'll work out some examples on the next homework. This is why radioactivity is so dangerous. Something decays (say a free neutron, which decays into a proton, a positron, and a neutrino). Since there is so much energy released in the decay, these particles come out with high energy, get into your body, and ionize atoms. This messes you up.

Splitting nuclei to release energy is called fission, while combining them is called fusion. Nuclear power plants use fission, the sun uses fusion. The original atomic bomb used fission of uranium, the hydrogen bomb uses fusion of hydrogen into helium. The key to making a fission bomb is to find a nucleus which allows a *chain reaction.* Such a nucleus can be split by hitting it with a neutron. When it splits, it not only releases energy, but it emits another neutron. This emitted neutron can in turn can split another nucleus, and so on. The trick is to get enough fissionable material together in one place before the whole thing blows up and ends the chain reaction. Bombs (luckily) are quite difficult to make. This is one reason. Another is that it is quite difficult to get a bunch of fissionable material. Most things which are radioactive are not fissionable. Natural uranium is mostly (99.3%) an isotope ^{238}U (92 protons and 146 neutrons). This is not fissionable, but only ^{235}U is, and this is only .7% of what you get when you dig uranium out of the ground. Plutonium (94 protons) is fissionable, but does not naturally occur, so it has to be made.