Modern Physics notes – Spring 2007
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Lecture 35

- Gravity and clocks
- Curved spacetime

- Born, chapter III (most of which should be review for you), chapter VII
- Fowler, "Remarks on General Relativity"
- Ashby on General relativity and the GPS, www.phys.lsu.edu/mog/mog9/node9.html

Gravity affects clocks

An experiment to test the effect of gravity on clocks was done in the early '60s by Pound and Rebka in a tower 20 feet from where my office was as a graduate student. First consider light shined *downward* in a freely falling elevator of height h. Inside the elevator, we're a happy inertial frame. We say it takes time $t = h/c$ to hit the bottom. We also say that there's no Doppler shift of the frequency of the light.

But how does this look from the ground? Say the light beam was emitted just as the elevator was released into free fall (i.e. at zero velocity). By the time the light hits the bottom, the elevator has accelerated to some velocity v. Since light travels so fast, the elevator isn't traveling very fast when the light hits the bottom, so v is pretty small, and we can use non-relativistic formulas for this (but not the light!). We thus simply have $v = gt = gh/c$.

Now let's see what this does to the frequency of the light. We know that even without special relativity, observers moving at different velocities measure different frequencies. (This is the reason the pitch of an ambulance changes as it passes you – it doesn't change if you're on the ambulance). This is called the Doppler shift, and for small relative velocity v it is easy to show that the frequency shifts from f to $f(1 + v/c)$ (it goes up heading toward you, down away from you). There are relativistic corrections, but these are negligible here.

Now back to our experiment. In the freely-falling elevator, we're inertial and measure the same frequency f at top and bottom. Now to the earth frame. When the light beam is emitted, the elevator is at rest, so earth and elevator agree the frequency is f. But when it hits the bottom, the elevator is moving at velocity $v = gh/c$ with respect to the earth, so earth and elevator must measure different frequencies. In the elevator, we know that the frequency is still f, so on the ground the frequency

$$f' = f(1 + v/c) = f(1 + gh/c^2)$$

On the earth, we interpret this as meaning that not only does gravity bend light, but changes its frequency as well.

If you start the light beam at the bottom, then the frequency measured on earth decreases – it is shifted to the red. Thus light escaping from a very massive star undergoes a *gravitational red shift*. This can be observed as well, by measuring the spectral lines from these stars and then comparing to those on earth. but Pound and Rebka measured this effect directly in the tower by my office. Note that with the $\sim 20\,m$ tower in a four-story building, this requires measuring a variation in frequency to about 1 part in 10^{15}. Now, using atomic clocks, this effect is easily measured.

So now forget the elevator – we've shown that light in a gravitational field shifts its frequency. This means that clocks at different altitudes measure time differently in the presence of gravity! This is like time dilation, except here nothing needs to be moving – the gravitational field does it. The time between ticks Δt is $1/f$, so

$$\frac{\Delta t_{sat}}{\Delta t_{earth}} = \frac{f_{earth}}{f_{sat}} \approx (1 + gh/c^2)$$

for a satellite at height h. The earth measures a shorter time between ticks than does the satellite. This means we on earth see the clock on the satellite as running *faster*. The standard clock in Boulder, CO (5400 feet) runs 6 microseconds a year faster than the clock at sea level in Greenwich, England. (Atomic clocks have an accuracy of about a microsecond a year.)

There is even a practical application of this result: the GPS consists of satellites in orbit with atomic clocks. Your GPS device gets signals from multiple satellites, each encoded with the information of what time and at what position the signal was sent. Since we know the speed of light, you can then figure out your exact position if there are at least three satellites. The speed of light is about one foot per nanosecond, so an error of one nanosecond is about one foot. On the homework, you'll work out by how much the clocks on the satellite change relative to the earth every day. The time you get from the clock thus must be corrected to compensate for this. It also must be corrected for the time dilation due to special relativity (the motion of the satellite – they go around the earth every twelve hours).

Spacetime is curved

So light is bent by gravity, and clocks at different altitudes run at different speeds. How do we generalize Lorentz transformations to account for frames accelerating with respect to each other, or, equivalently, in a gravitational field? The whole story is called general relativity, and to really quantitatively understand what's going on requires a fair amount of mathematical sophistication. The area of math involved is called *differential geometry*, and what it involves is understanding how to do calculus on surfaces which are *curved*, such as the surface of a sphere.

So why do we need to understand calculus on surfaces which are not flat? The fact that light curves means that the quickest way of getting from one point to another in a gravitational field is not a straight line! Thus if there is gravity present, we can't use rigid rods to measure a distance or a single clock to measure a time interval the way we did in special relativity. Spacetime itself is curved.

Why is space time curved? We can apply these ideas to a merry-go-round in outer space. In the frame of the merry-go-round, you feel a centrifugal force outward: it's $m\omega^2 R$. If you don't know about the motor of the merry-go-round, you can't distinguish that from the possibility is that you're in a horizontal gravitational field of strength $\omega^2 R$ pointing way from the merry-go-round (say coming from some enormous asteroid). This is analogous to, but reversed from, free fall. Both frames agree that the other accelerating with respect to them. One frame in each case says there's gravity. But here it's the merry-go-round which says there's "horizontal gravity" (the centrifugal force). With the elevator in free fall, they say there's no gravity, while the ground says there is.

Now let's measure the ratio of the circumference of the merry go round. On the ground (here the inertial frame), we place many short sticks of fixed length L around the edge of the merry go round, and radially from the center. Since the sticks are straight, this gives a slight error, but as we increase the number of sticks, the error decreases. Denote N_{edge} the number of sticks around the edge, and N_{radial} the number of sticks pointing radially. As we use a larger number of shorter sticks, we find of course

$$\frac{N_{edge}}{N_{radial}} = \frac{N_{edge} L}{N_{radial} L} = \frac{C}{R} = 2\pi$$

Since the ground is an inertial frame, we interpret this as meaning that the ratio of the circumference C to the radius R is 2π as Euclidean geometry says. Now let's interpret this from the merry-go-round. Because each point on the edge is moving with some speed $v = \omega R$ with respect to the sticks on the ground, people on the merry-go-round see each stick of length L contracting, so that they measure $L' = L/\gamma$. The radial sticks, however, are still measured at the same length L, because the motion is perpendicular to them. The merry-go-round and people on the ground of course agree on the number of radial sticks N_{radial} and the number of sticks N_{edge} around the circumference (an integer can't change between frames!). What they do not agree on is the length of the circumference. They would say that the circumference is $C' = N_{edge} L'$, so that $C' = C/\gamma$. Comparing with before.

$$2\pi = \frac{N_{edge}}{N_{radial}} = \frac{N_{edge} L}{N_{radial} L} = \frac{C'}{R'\gamma}$$

Thus on the merry-go-round, we have for a circle $C = 2\pi\gamma R$. This is not Euclidean geometry!

But from the equivalence principle we know that this the merry-go-round can be interpreted as a frame at rest in a gravitational field. Thus gravity bends space. You can come up with arguments like this for time as well. This means that in the presence of gravity, spacetime is curved!

As no doubt you know, once you're on a curved space like a sphere, measuring things gets trickier. For example, if I draw a circle on the surface of a sphere, it's obvious that $C \neq 2\pi R$ if we define all distances to remain on the sphere. The Pythagorean theorem fails as well.

First, let me give a more precise definition of what curved is. Take 12 equal-length rods, and arrange them in a hexagon with spokes. If you can do this on some surface, we say that the surface is flat at this point. But on say on the top of a mountain, one rod is too long. That's called "positive" curvature. But on a saddle, one of the rods isn't long enough. That's called negative curvature.

So how do we quantify this? As I said, to really do this right requires a lot of math. But some useful facts from special relativity show at least how to set things up. We learned there that already things are complicated: length contraction and time dilation occur even without acclerating frames. But we did learn the interval Δ is the same in any frame. Thus without gravity, we can say that $\sqrt{\Delta}$ is the "distance" between two events in spacetime. What this means is that for two space like events, $i\sqrt{\Delta}$ is the distance between the two events in the frame where they are simultaneous. For two timelike events, $\sqrt{\Delta}$ is the time interval between the two events in a frame where the events are at the same place (i.e. the "rest" frame).

In special relativity, Δ is like a distance, except for the funny minus sign. Say that two events are close together in space so that $dt = t_2 - t_1$ and $d\vec{x} = \vec{x_2} - \vec{x_1}$. We say that the distance $d\Delta$ between two events close together in space time is

$$d\Delta = \sqrt{c^2(t_2-t_1)^2 - (x_2-x_1)^2 - (y_2-y_1)^2 - (z_2-z_1)^2} = \sqrt{c^2(dt)^2 - \sum_{i=1}^{3} dx_i^2}$$

where $dx_2 = dy$, etc. We say that this spacetime (no gravitational fields) is *flat*. Roughly speaking, it means that the coefficients of the terms on the right-hand-side do not depend on space or time. Note that this is basically the Pythagorean theorem with minus signs.

So this sounds horribly complicated. But here's the rule. Even on curved surfaces, there is still a notion of "shortest path". For example, on the earth, the shortest path follows great circles. To get from here to Korea, you fly over the Arctic Ocean and Siberia. (Really!) A great circle is the edge of a disc slicing the earth in half. To measure the shortest distance in a curved spacetime, there is a formula for $d\Delta$ but now where the coefficients depend on space and maybe time too. To write down a nice equation, let $x_0 = ct$, so $dx_0 = cdt$

$$d\Delta = \sqrt{\sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g^{\mu\nu}(t,\vec{x})dx_\mu dx_\nu}$$

where the 16 functions $g^{\mu\nu}$ are called the *metric*. If there is no curvature (i.e. no gravity) then we have $g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, and $g^{\mu\nu} = 0$ if $\mu \neq \nu$. Now we can determine the shortest path simply by finding Δ for all paths between the two events: the one with the smallest $|\Delta$ is the shortest one. This path is called a *geodesic*. (I have no idea why Born calls it a "geodetic".)

Sorry for the mathematical diversion. Here's what it has to do with gravity. *An object in free fall follows geodesics, i.e. the shortest path in spacetime.* By "free fall" we mean in the absence of all external fields. Thus even in the presence of gravity, "free" particles follow the shortest path in spacetime! When gravity is present, these paths seem curved to a observer who feels gravity's force. (i.e. some one not in free fall). In special relativity, the shortest paths are straight lines. Once the coefficients in the above equation for $d\Delta$ (the metric) start to depend on space and time, this is no longer true.

To reiterate: we have said if there is no gravity, then everything reduces to special relativity: spacetime is flat. But of course there is only no gravity if there is no matter: a world of light only is not particularly interesting. When there is gravity, then light no longer travels in straight lines. The more matter, the more gravity, and the more curving things do.