Modern Physics notes – Spring 2007

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Lecture 36

- Einstein's equations
- Black holes
- Dark energy and the cosmological constant

- Weinberg, chapters I and II
- Fowler, "Remarks on General Relativity"

Einstein's equation

So we haven't said one crucial thing. How do we know *how* spacetime is curved? In other words, how do we know what the metric is? This determined by *Einstein's equation*. I can't write this out precisely without using some differential geometry, but I can explain it intuitively. If you know the metric, then there is a precise mathematical way of defining the curvature of spacetime. Einstein's equation is then of the form

curvature of a spacetime point = energy and momentum density at that point

This boils down to a differential equation for the metric. Solving this equation tells you what spacetime is like. It's the equation whose solution tells you the shape of the universe!

One kind of solution to Einstein's equation is a *gravitational wave*. This is just as it sounds. Just like if a light wave hits you, you feel an electromagnetic force, if a gravity wave hits you, you feel a gravitational force. So far, gravitational waves have been observed only indirectly, in binary neutron stars. A binary star consists of two stars orbiting each other. Just like an accelerating electrical charge radiates light waves, an accelerating mass radiates gravitational waves. You may remember that this is why the atom is unstable classically – it emits radiation, loses energy, and will crash into the nucleus. The same goes for binary stars. But since gravity is so weak a force, it's barely perceptible. But as the two get closer together, their angular speed goes up, and this has been measured. The rate at which the two are getting closer are in accord with the result you get from solving Einstein's equation.

There's a big experiment under way (called LIGO, Laser Interferometer Gravitational-Wave Observatory) to measure gravitational waves directly.

Black holes

Another striking consequence of Einstein's equation is that there are "black hole" solutions. This means that it is possible for there to be so much mass in a small enough area to mean that even light can't escape the enormous gravitational field. In fact, there is a "singularity" at the location of the black hole: essentially, our classical notion of spacetime itself ceases to exist! Perhaps quantum mechanics removes this singularity, but since no theory of quantum gravity is fully understood (string theory is at the moment the only possible candidate), we don't know.

Black holes very probably exist; in fact, there is probably one at the center of the Milky Way. Detecting them is somewhat difficult, for obvious reasons. But stars close enough to them will orbit around them, and from that we can determine the mass of the black hole. The one at the center of the Milky Way seems to have a mass of around 4 million solar masses.

Obviously, without writing down Einstein's equations explicitly we can't be too precise about black holes. But we can learn one interesting thing just by dimensional analysis: the "size" of the black hole. The *event horizon* surrounding the black hole is the distance from the center where once an object falls in, it can never escape. This is where the escape velocity is greater than the speed of light. You can guess the radius of the event horizon by dimensional analysis. The only relevant physical quantities in the problem are the mass of the black hole, Newton's gravitational constant G, and the speed of light c. In terms of mass $[M]$, length $[L]$ and time $[T]$ G has dimensions $[L^3/MT^2]$. The radius of the event horizon R_e has no mass in its units, so it must depend on G and M as GM, which has dimensions $[L^3/T^2]$. The only way to combine this with c (dimension $[L/T]$) to end up something of dimension $[L]$ is to divide by c^2. Thus up we know that $R_e = xGM/c^2$ for some dimensionless number x. It turns out this number is two, so the radius of the event horizon for a black hole of mass M is

$$R_e = \frac{2GM}{c^2}$$

Let's drop something into a black hole. The interesting thing is that we never see it get to the event horizon! The point is that as we watch an object fall, we see its clock slow down. At the event horizon of a black hole, the gravitational field is so strong that we see the clock stop altogether!

Since we don't have a fully-understood theory of quantum gravity (string theory is the only candidate at the moment), we don't fully understand black holes. But there's one amazing prediction of Hawking's: black holes radiate particles! One thing we know about particle accelerators is that at high enough energies, it's possible to create particle-antiparticle pairs. Since gravitational fields are so large near a black hole, there's enough energy to create these pairs. Say the pair was created by the event horizon, and one of the two fell in the black hole, and the other escaped. Thus the black hole can emit particles! By energy conservation and $E = Mc^2$, this means that the black hole is losing mass. Eventually it can radiate away enough mass so that it will no longer be a black hole – it evaporates. Since we're not even positive we've seen a black hole, we're not likely to sees a black hole evaporate any time soon.

The cosmological constant

When Einstein wrote down his equation, he realized right away that its solutions were the large-scale structure of the universe itself. By "large-scale", I mean we don't worry about unimportant local things like us, planets, stars, or even galaxies. On a large enough scale (something like a billion light years), the universe is *homogeneous* (independent of position, i.e. invariant under translations). In other words, if you take large enough boxes, you will find roughly the same number of galaxies in any box. Another piece of evidence of homogeneity is the fact that the cosmic microwave background is extremely homogeneous, an issue we'll come back to. The universe is also *isotropic* (independent of direction, i.e. independent of rotations).

Einstein thus looked for solutions to his equations which are both homogeneous and isotropic. That's good. He then also looked for solutions which are *static* (no relative motion). He found, probably to his disappointment but not to his surprise, that there weren't any static solutions. This isn't surprising because gravity is always an attractive force. Thus if you start with a bunch of static objects feeling gravity, they will begin to fall in toward each other. (The quantum effects which stabilize the atom aren't important on cosmological scales now, although they probably were important in the early universe.)

Now we know that the universe is not static, and is in fact expanding. We'll talk about that much more in the next lectures. But at the time, Einstein tried to get a static solution. He realized that it was possible to extend general relativity and include a new force which stabilized the universe against gravitational collapse at large scales. He added another term to his equation, so it was of the form

$$\text{curvature of a spacetime point} = \text{energy and momentum density at that point} + \Lambda g_{\mu\nu}$$

where $g_{\mu\nu}$ is the metric at that spacetime point, and the coefficient Λ is called the *cosmological constant.*

Within the next 15 years (circa 1930), the expansion of the universe became experimentally clear, so including this extra term became unnecessary. Actually, it was wrong anyway – even though the cosmological constant does allow for a static universe, it is an unstable solution. A small perturbation will either result in collapse or expansion. Einstein referred to the introduction of the cosmological constant as his "greatest blunder". But just in the last 5 years, it's been established from a variety of measurements that a non-zero Λ is the most likely explanation for the "dark energy" in the universe!