Modern Physics notes – Spring 2007
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**Lecture 37**

- The red shift
- The Hubble constant
- Critical density

- Weinberg, chapters I and II
- cosmological parameters: Spergel et al, http://arxiv.org/abs/astro-ph/0603449

**Expanding universe**

So how do we know the universe is expanding?

There are two key pieces of physics behind this observation. The first is the Doppler shift, the second is the existence of quantized energy levels. Non-relativistically, the Doppler shift is

$$\frac{\lambda_{obs}}{\lambda_{emit}} = 1 + \frac{v}{c}$$

for a signal emitted by an object moving at velocity $v$ away from you. You can look in the "Mathematical Supplement" of Weinberg for a derivation, if you forget. On the homework, I'll make you derive the relativistic version.

The key point is that the Doppler shift relates a shift in frequency to the relative velocity. Thus if you measure a frequency shift, you know the velocity. This we do because of a fundamental fact of quantum mechanics: atoms emit photons of quantized frequencies. As long as the atom is the same here as in another galaxy, we know that the frequency in their frame is that same as that of an atom of the same type on earth. But because of the Doppler shift, we'll measure a different frequency. By comparing the two, we know the Doppler shift, and hence the relative velocity.

In cosmology, it is conventional to define the parameter $z$, which is the fractional increase in wavelength:

$$z = \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}}$$

For a given object, $z$ positive is a red shift, $z$ negative is a blue shift. By comparing to the Doppler-shift formula, we see that non-relativistically, $z = v/c$. Thus $z$ positive means that the objects are moving apart; $z$ negative means they're moving together. They now do red-shift surveys with hundreds of thousands of galaxies, and almost all of them have $z$ positive: the universe is expanding!

Note that the non-relativistic formula for $z$ has a maximum value of 1. In recent years, supernova have been observed with $z > 1$, the maximum of about 1.7. We'll see later that this means its light came from about 11 billion years ago.

Now we can be precise about the word "expand". Usually when we use that word, it means that something is of finite volume, and that volume is getting bigger with time. Here we don't need that notion (in fact the universe is almost certainly not of finite volume). What we mean by expanding is that the average distance between galaxies is decreasing. (Almost) everything is moving away from us.

**The Hubble constant**

Now that we know the universe is expanding, the first obvious question to ask is: what's the center? Usually when we think of something like a balloon being blown up. This obviously has a central point which is not moving. But if the expansion has a center, then our assumption of homogeneity is obviously wrong: the center is a special point.

There's only one way of having an expansion and homogeneity at the same time. This is if the relative velocity between two objects is proportional to the distance in between them. By looking at the figure in Weinberg, you can can see how this works. Everybody agrees that the relative velocity of two objects is proportional to the distance between them, as given by the formula

$$v = Hd$$

Using the non-relativistic Doppler shift formula allows us to relate this to the red shift:

$$d = \frac{zc}{H}$$

The constant $H$ is called the "Hubble constant". It is constant in *space*, not in time. For example, ignore all forces like gravity. Then the relative velocity between two objects will remain constant, even though $d$ is increasing. Thus $H$ will go down with time.

Without knowing the value of $H$, this allows us to determine relative distances from the redshifts: twice the $z$ means twice the distance (and twice as old, because the light will have taken twice as long to reach us). Note this is an *average* relation: a given galaxy will be moving in some random direction as well, changing the value of $z$. For Andromeda, $z$ is negative, but that does not mean it is a negative distance away. But now, they have seen hundreds of thousands of galaxies with $z$ large enough so that we can neglect this random motion. For example, in one sky survey I looked up they have thousands of galaxies with $0.01 \leq z \leq 0.17$.

The much harder thing to do is to determine absolute distance. In other words, what is the Hubble constant $H$? To do this, we need to figure out the absolute distance to something. Then we can fit $H$ by using this formula, because we measure $v$ by the Doppler shift and $d$ directly. The distance to close stars can be found by using *parallax*: you measure the position of the star, and then wait six months and measure it again. The earth is in a different position, and from the two angles and one distance you've measured, you can complete the triangle and find the distance to the star. Unfortunately, all the stars close enough to do this on are in our galaxy, which means they're not expanding with respect to us. But this led the way to an important observation (by Henrietta Levitt) that a certain kind of star, called a Cepheid, has an unusual property. Its brightness (luminosity) oscillates in time, and (the important part) the period of oscillation is proportional to its absolute luminosity! Knowing the absolute luminosity of something allows us to get the absolute distance. You measure how bright it is here, and knowing that ower falls off as $1/r^2$ you then can find the distance (assuming there's no intergalactic dust that absorbs the light, etc).

This was the first way of measuring absolute distance cosmologically, and wasn't very accurate: the original number Hubble used was off by a factor of 10 or so. Even when I was a graduate student, the Hubble constant was known only to about a factor of 2. But just in the last 5 years, the data have gotten much better. Now there are a variety of "standard candles" like the Cepheids. One standard candle is a supernova: it turns out because all supernovas behave in the same way, one can figure out what their absolute luminosity and therefore absolute distance as well. The current best-fit value Hubble constant is

$$H = 22.5 \pm 1 \frac{km/s}{\text{million light years}}$$

If we assume that the galaxies aren't accelerating or decelerating, we can use the Hubble constant to estimate the age of the universe. We do this by tracing backwards in time to where there was no distance between galaxies. If a galaxy is a distance $d$ from us and traveling at a constant relative velocity $v$, that means a time $t = d/v$ in the past, it was on top of us. This time was the time the Big Bang happened. From our Hubble formula, this means the age of the universe at approximately $t_{universe} = d/v = H^{-1}$. Putting the Hubble constant in more useful units for this purpose, we get

$$t_{universe} = H^{-1} = \frac{10^6 ly}{22.5 km/s} \frac{1 year \times 3.0 \times 10^5 \, km/sec}{ly} = 13 \text{ billion years}$$

This is only an approximation because the velocity is not constant over time – gravity causes a deceleration (it is always attractive), and the cosomological consant causes acceleration. By luck it agrees with more accurate methods, which give the age of the universe to be $13.7 \pm .2$ billion years.

### Critical density

From classical physics and black holes, you're familiar with the concept of escape velocity. You fire an object upward: if it's going fast enough, it can escape, otherwise it falls back down again. With a black hole, the escape velocity becomes the speed of light at the event horizon, so nothing can escape.

For a single object of mass $m$ on the surface of the earth, it's easy to use conservation of energy to compute the escape velocity.

$$E = \frac{1}{2}mv^2 - \frac{GmM}{R}$$

Now if $E < 0$, the object can't escape. This is because as it gets farther and farther away, the potential energy gets smaller and smaller. By conservation of energy, $v$ must get smaller and smaller. Eventually $v = 0$, and the object must turn around and go back. For $E > 0$, the object can escape. Thus for $v = v_c$, we have $E = 0$, so

$$v_c^2 = \frac{2GM_e}{R_e}$$

Ironically, for black holes, this gives the right answer if you just substitute $v_c = c$: light can't escape. This is a complete coincidence, because the formulas here are non-relativistic (you can't use $1/2mc^2$ for the kinetic energy of light!)

Now let's think about escape velocity in terms of the universe. The universe is expanding, but gravity is an attractive interaction. This is why Einstein couldn't get a static universe: the gravity will make it collapse. Thus there are three possibilities regarding the expansion of the universe: either gravity wins, expansion wins, or it's a tie. In other words, the presence of matter (without a cosmological constant) causes a deceleration. The issue is whether the deceleration is large enough to eventually cause the universe to stop expanding and begin to contract. This is called a *closed* universe. If the deceleration is too small and the expansion goes on forever, it is called an *open* universe.

The amount of deceleration is controlled by the density. To see this, let's redo our escape velocity computation in the context of the universe. There's a useful fact about inverse-square law forces. In a situation with spherical symmetry (which applies here because of the isotropy), forces from outside the sphere all cancel, while forces from inside the sphere average out so that they act from the center of the sphere. We use this all the time: the fact that $g = GM/R$ follow from this: we don't have to integrate over the whole earth (if you did the integral, you'd just get this!). If you go down into the earth, you feel the effects only from the earth beneath you. This means force of gravity weakens if you tunnel down into the earth, because inside this sphere of radius $R$, there is mass $M = \rho V = 4\pi R^3/3$. Now let's consider two galaxies in the universe a distance $R$ apart. Draw a sphere with one galaxy at the center and the other on the surface (it doesn't matter which one, because of homogeneity). This sphere is of course of radius $R$. Now we also know from the Hubble relation that the relative velocity is $v = HR$. Thus in the frame where the galaxy at the origin is at rest, you get

$$E = mR^2 \left( \frac{1}{2} H^2 - \frac{4}{3} \pi G \rho \right)$$

In determining whether $v$ is larger or smaller than the escape velocity, only the sign of $E$ matters. Since the only place $R^2$ appears is in front, the distance $R$ between the two galaxies doesn't affect the sign of $E$. This is a nice consistency check: we would violate homogeneity if the escape velocity depended on distance.

So now we can define the *critical density* $\rho_c$ of the universe. If the density $\rho$ is greater than $\rho_c$ the expansion of the universe will eventually cease. For $\rho < \rho_c$, the expansion goes on forever. These cases correspond to $E < 0$ and $E > 0$ in our sphere. Thus the critical density is related to the Hubble constant:

$$\rho_c = \frac{3H^2}{8\pi G}$$

Current experimental evidence shows to great accuracy that the universe is indeed at the critical density. One test is that the universe is *flat*. When the universe is at critical density, it is also flat, in the sense we discussed last time. This is experimentally testable by looking at two different very-distant objects. Basically, one measures the angles between the three objects: if they add up to 180 degrees, the universe is flat!

If the universe is at critical density, eventually all galaxies will be far enough from each other so that we can ignore gravity. Then, if there are no other forces, $v$ is a constant, and $d$ is increasing linearly with time. This means $H \propto 1/t$ eventually. But there seem to be other forces...