Modern Physics notes – Spring 2007

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Lecture 5

- Uncertainty principle

- Feynman, 1,8, 2.2, 2.6
- Fowler, "More on the Uncertainty Principle"

Uncertainty principle

It's now worth spending some time trying to understand how the things we've discussed are consistent. In particular, we have the bizarre fact that when interference occurs, we don't know if the electron went through slit 1 or slit 2. If we shine short-wavelength light on it, we can tell which slit, but this destroys the interference pattern: photons with $\lambda > d$ have momentum greater than that of the electrons, and so drastically change the experiment. So in this experiment there seems to be no way of telling whether the light went through slit 1 or slit 2. The key question is: can you figure out a different experiment that will answer this question?

The answer, quite remarkably, is no. No experiment will ever answer the question of which slit the electron goes through when there is interference. But let's try to understand the reason why we can't ask it.

This inability to answer every question one can pose can be turned into a precise relation, called the uncertainty principle. We don't have the tools in this class to prove this relation directly, but there's a nice way of illustrating it. This comes from doing another interference experiment, but this time with a single slit of width W. Consider a beam of electrons, all of which have the same momentum p_0 in the x direction. Put the source of electrons very far away from the slit, so that the electrons originally have y momentum $p_y = 0$. (Any electrons which start with $p_y \neq 0$ won't hit the slit.)

Let's look at this experiment both from the point-of-view of waves scattering, and of particles scattering. First, the waves. In the double-slit experiment, we had interference when the distance between slits was roughly the same size as the wavelength. In a single-slit experiment, one gets interference if the width W is about the same as the wavelength. This isn't hard to understand: think of the slit as an extended source of the wave. The wave landing at a given point on the backstop is made up of light traveling from all the points on the slit. Since light coming from two different points in the slit has traveled different distances, it can interfere.

To get the full answer one sums up (integrates) the waves coming from everywhere on the slit. That's some work, and we don't need the precise answer anyway. What we need is to understand just where the first minimum approximately is. Remember that two waves destructively interfere when the difference d_1 and d_2 differs by a half-integer times the wavelength, the smallest distance being

$$d_1 - d_2 = \frac{\lambda}{2}$$

For a single slit, you get destructive interference when d_1 is the distance from the center of the slit, and d_2 is the distance from the edge of the slit. When this is true, roughly speaking, every point on the slit cancels with another point where the distance differs by $\lambda/2$. By drawing a picture, one can find the angle θ_{min} between the original x-direction of the waves and the direction of the minimum. It is given by

$$\sin\theta_{min} = \frac{\lambda/2}{W/2} = \frac{\lambda}{W}$$

That's pretty standard – one would get the same answer for light waves, and for electron waves. But I've emphasized all along that electrons are still particles: they still occur one-by-one. Now we know light consists of discrete lumps as well. So let's understand what this experiment means in particle language. Remember our interference plot is for the *probability*: we don't know precisely where the electron will land. Thus a given electron will travel off at some small angle θ, which we don't know exactly. We do know, however, that the probability falls off as we get farther and farther away from the middle, i.e. as θ gets larger. In other words, the electron will probably land in the regions between the first few minima. Thus up to factors of 2, the uncertainty in θ is

$$\Delta(\sin\theta) \approx \sin\theta_{min} = \frac{\lambda}{W}$$

We can now see what the consequences of this are for the momentum. Initially, we know that the particles (whether they be photons or electrons) have $p_y = 0$. But because of scattering off the slit, they get a y momentum. Indeed, if you look at the interference pattern, you see that the particles extend beyond where they would have landed had they gone straight through the slit. So the scattering off the slit must give them some vertical momentum. Since for a given electron, we don't know precisely what its y momentum will be, just like we didn't know what θ was. However, we do know that for an electron traveling off at angle θ, its momentum must be

$$p_y = p_0 \sin\theta$$

if we make the very reasonable assumption that $p_y \ll p_0$, so that the total momentum remains $\approx p_0$. Thus the uncertainty in θ translates to an uncertainty in p_y:

$$\Delta p_y \approx p_0 \Delta(\sin\theta) = p_0 \frac{\lambda}{W}$$

Now let's combine results from the wave and particle pictures. In particular, let's use the crucial relation of the wavelength of a particle to its momentum, namely $p = h/\lambda$. Since the total momentum doesn't change much, $p \approx p_0$, and we can set $p_0 = h/\lambda$. Plugging this in relates the uncertainty in y momentum to the width of the slit, i.e.

$$\Delta p_y \approx \frac{h}{W}$$

Thus in this experiment, we can predict the final y momentum only up to an uncertainty h/W.

As we talked about in great detail in the double-slit case, there's another uncertainty in our measurement as well. This is in *where* the electron came from. Before, it was which slit the electron came from. Here, the uncertainty is in where in our one slit the electron came through: all the arguments we made in the double slit case can be made here. All we know is that electron came through the slit of width W. Thus the uncertainty Δy in the y-position is W:

$$\Delta y = W$$

Putting our two relations together gives

$$\Delta y \Delta p_y \approx h$$

This is striking: note that this no longer depends on the size of the slit or the wavelength of the light. It only depends on Planck's constant. The narrower the slit gets, the larger the uncertainty in momentum.

So we now have a precise statement of the uncertainty for this system. The absolutely incredible prediction of quantum mechanics is that there exists such a relation for **any** system. **One can never predict both the momentum and the position at a given time with arbitrary precision**. They always obey the *Heisenberg Uncertainty Principle.* For any direction, we have the same relation, e.g.

$$\Delta x \Delta p_x \geq \frac{h}{4\pi}$$

It means that if we know a position of a particle very accurately, we can't predict its momentum very accurately (so it will be moving very fast). If we know instead the momentum very accurately, then we can't predict where it will go very accurately.

Of course h is a very small number in MKS units, so we don't notice this in the everyday world: the uncertainty is very small. But when starts looking at individual particles with very small mass (e.g. $9.11 \times 10^{-31}\,kg$ for an electron), and their behavior at very short distances, quantum effects like this are all-important.

So the uncertainty principle tells us we can't know everything. Many philosophers have made a great deal out of this; it certainly is an amazing fact. It also made many people unhappy, including Einstein. He of course acknowledged that quantum mechanics explained the experimental data accurately, but always believed that there would be some underlying theory which could predict everything (at least in principle – everyone agrees this is mainly a philosophical issue). In fact, he, Podolsky and Rosen pointed out there was a bizarre effect of quantum mechanics which seems like doing a measurement in one place will effect a measurement in another place, arbitrarily far away. It turns out this "paradox" has now been experimentally observed! We'll discuss that in more detail later. But even without this experimental result, you can prove mathematically that, roughly speaking, if quantum mechanics is effectively correct (i.e. you can use it to explain the data), then there can **not** be some underlying theory where you know everything. And since every day we all test quantum mechanics by verifying that essentially everything electronic works, we can safely say that quantum mechanics is effectively correct. This uncertainty is the way of the world.

So the uncertainty principle seems to be a negative result: it tells you what you can't know. However, it is also a quite useful for giving new results, by using hand-waving arguments. Next we'll use it to determine the size of the atom.