Modern Physics notes – Spring 2007
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Lecture 7

- Size of the atom
- Three rules of quantum mechanics

- Feynman, 2.4-2.6, 3.1

The size of the atom

Let's follow Feynman and use the uncertainty principle to estimate the size of an atom. Say the electron's allowed orbits extend out to a some radius a. If we do experiments, we'll see that the electron is in a different position each time we look. So its uncertainty in position Δx is about a. By the uncertainty principle, we then know that the uncertainty in momentum is about $h/(2\pi a)$. (All of these numbers can be multiplied or divided by factors of 2 or 3 or π: what we are really doing is dimensional analysis combined with quantum mechanics). I kept the 2π there because a factor of 6 is not that small, but really because it is often more convenient to write things in terms of $\hbar$, which is defined as

$$\hbar \equiv \frac{h}{2\pi} = 1.055 \times 10^{-34} J \cdot s$$

The symbol $\equiv$ means "is defined as". One reason physicists like to use $\hbar$ is that it's easier to say "h-bar" than "Planck's constant".

The fact that uncertainty in momentum is $\Delta p \sim \hbar/a$ means the momentum itself is probably around $\hbar/a$. If the momentum were larger, we could then do scattering experiments with light of momentum larger than $\hbar/a$ without disturbing the electron. We could then determine its position to better accuracy then a, contradicting our original assumption that $\Delta x \sim a$. If its momentum is about $\hbar/a$, then its kinetic energy is about $(\hbar/a)^2/2M$. (By the way, Feynman was a little sloppy. He writes h in the text, but when he gets the numbers out, he uses $\hbar$.)

So now let's look at the energy of the hydrogen atom. The charge on the nucleus is just $-e$, and the charge of the electron is e. (We define e to be negative; the answers will depend only on e^2, so this definition doesn't affect the results, a useful check.) In addition to the kinetic energy, there's an electrostatic potential energy coming from the two charges being a distance a apart. The total energy is therefore about

$$E = \frac{\hbar^2}{2Ma^2} - \frac{e^2}{4\pi\epsilon_0 a}$$

The minus sign in the second term is because the electron and the nucleus are attractive. Notice that as a decreases, we lower the potential energy, but we also increase the kinetic energy (the uncertainty principle means that cramming a particle into a smaller box increases its energy).

The value of a is determined by finding the minimum of this energy The reason is that in the absence of any external sources of energy, it's possible to get rid of energy by say emitting light, but there's no way of getting new energy. Thus the system will settle down into its lowest energy state, and then be stable. To find this minimum value of energy, we find where $dE/da = 0$. Precisely, at the minimum $a = a_0$, we have

$$0 = \left.\frac{dE}{da}\right|_{a=a_0} = -\frac{\hbar^2}{m_e a_0^3} + \frac{e^2}{4\pi\epsilon_0 a_0^2}$$

Solving for a_0 gives

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \frac{(1.055 \times 10^{-34})^2}{(8.988 \times 10^9)(9.106 \times 10^{-31})(1.602 \times 10^{-19})^2} m = .529 \times 10^{-10} m$$

So the size of the hydrogen atom is about $10^{-10}\,m$, which is called an angstrom. (People today tend to use $nm = 10^{-9}m$ instead of angstroms.) Using a simple hand-waving argument, we know the approximate size of an atom! (note that this factor of 2π Feynman is sloppy on changes the answer by a factor of 40, because it's squared!)

To find the energy of this electron at this minimum, we can now plug our value of a_0 back into the expression for E. After a little algebra, we get

$$E = -\frac{e^2}{8\pi\epsilon_0 a_0} = -13.6\,eV$$

The negative energy means that the electron has less energy when it is in the atom than it has if it is not in the atom ($E = 0$ when $a \to \infty$). This is good, because it means that electrons want to stay in atoms, instead of moving away! This number $13.6\,eV$ is called a Rydberg. It turns out that we've been lucky. We've ignored factors of two and pi and so forth, so there's no reason this should be the exact answer. However, for the hydrogen atom this is the correct **binding energy**: it takes one Rydberg of energy to pull the electron away from the hydrogen atom.

By the way, note that Feynman didn't forget the $4\pi\epsilon_0$: he's working in units of charge where this is 1. In these units (called stat-coulombs), the charge of the electron is $1.6 \times 10^{-19}\sqrt{9 \times 10^9}$ stat-coulombs, so you of course get the same answer at the end.

So quantum mechanics explains why atoms are stable. If the electron were to spiral into the nucleus, Δx would be smaller, and the uncertainty principle would require that the momentum be larger. This increase in kinetic energy balances with the decrease in potential energy right at a_0, as we've shown.

Quantum mechanics also explains why we don't fall through the floor. The atoms in our shoes push against the atoms in the floor. When the atoms are squashed into a smaller space, their momentum and hence energy must increase, because of the uncertainty principle. Remember that energy wants to be minimized, so the force resists the compression. Classically, the squashing would lower the potential energy, but without the uncertainty principle, there would be no corresponding increase in kinetic energy.

Three rules of quantum mechanics

We're now going to dump the handwaving and set things up more systematically. In particular, to get any further, we need to understand the probability amplitude/wavefunction better.

In chapter 3, Feynman discusses three general principles of quantum mechanics. The first is:

> The probability something will happen is proportional to the magnitude squared of a complex number called the *probability amplitude*.

It is convenient to introduce some new notation for this. Consider the probability amplitude for a particle emitted at s will arrive at x. Our new way of writing this is as

$$\langle x|s\rangle$$

The way to think of this is to read right to left. A particle is emitted at s, and this is represented as $|s\rangle$. This is called the *initial state*. The particle is observed at a point x, and that is represented as $\langle x|$. This is called the *final state*. Putting the two together gives the amplitude $\langle x|s\rangle$, a complex number.

This seems a very fancy way of writing a single complex number. It is useful to do it this way when we start to combine amplitudes. Remember that the reason we got interference was because one adds together the amplitudes, not the actual probabilities. This is Feynman's second principle:

> When a particle can reach a given final state by two possible routes, the total amplitude for the process is the *sum of the amplitudes* for the routes considered separately.

For the double-slit experiment, this means that

$$\langle x|s\rangle_{\text{via 1 or 2}} = \langle x|s\rangle_{\text{via 1}} + \langle x|s\rangle_{\text{via 2}}$$

This is our earlier formula $\phi_{12} = \phi_1 + \phi_2$ for interference written in new notation. The probability it will end up at x is $|\phi_{12}|^2 = |\langle x|s\rangle|^2$, the magnitude squared.

Now we want to understand how to combine amplitudes in the case that one event happens after another. (An "event" is something which happens at a particular place at a particular time.) The third principle is

> When a particle goes by some particular route, the amplitude for this route can be written as the *product* of the amplitude to go part way with the amplitude to go the rest of the way.

For our double-slit experiment, this means that we can define the amplitude that the particle go from s to slit one as $\langle 1|s\rangle$, and for it to go from slit one to x as $\langle x|1\rangle$. The third principle means that

$$\langle x|s\rangle_{\text{via 1}} = \langle x|1\rangle\langle 1|s\rangle$$

We can do the same for slit 2. This lets us rewrite the interference as

$$\langle x|s\rangle_{\text{via 1 or 2}} = \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle$$

So now this notation doesn't look so bizarre. It gives us an easy way for combining amplitudes for more complicated process. Consider the more complicated process drawn in Fig. 3-2 of the book. There are six different ways a particle can get from s to x. We thus have

$$\begin{aligned}\langle x|s\rangle &= \langle x|a\rangle\langle a|1\rangle\langle 1|s\rangle + \ldots \\ &= \sum_{j=1,2}\sum_{\alpha=a,b,c}\langle x|\alpha\rangle\langle\alpha|j\rangle\langle j|s\rangle\end{aligned}$$

You should think of $1, 2$ and a, b, c as being *intermediate states*. We don't have any way of observing where the particle is at these stages without disturbing the experiment, so we must sum over all intermediate possibilities.