Modern Physics notes – Spring 2007
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Lecture 8

- The amplitude for a free particle
- The particle in a box, again

- Feynman, 3.1-3.2

A free particle

To understand interference in this new way of writing things, we need to know the probability amplitude for a particle to get from one point to another. In general, this of course will depend on the interactions in the system (e.g. the electrostatic attraction in the hydrogen atom). For now, let's focus on the case where the particle is "free", which means that it has no interactions (no external forces on it). In general, we represent a point in three dimensional space as a vector $\vec{r}$ of the three coordinates x, y, z. Then the probability amplitude for the particle of definite momentum $\vec{p}$ to get from $\vec{r}_1$ to $\vec{r}_2$ is

$$\langle \vec{r}_2 | \vec{r}_1 \rangle \propto e^{i\vec{p}\cdot(\vec{r}_1 - \vec{r}_2)/\hbar}$$

This equation explains precisely how a "particle" acts like a "wave". Remember that e^{ia} is periodic under a shift $a \to a + 2\pi$. This means the amplitude is periodic under a shift $\vec{p}\cdot\vec{r} \to \vec{p}\cdot\vec{r} + h$. A wave has the same property. Thus waves in quantum mechanics are "probability amplitude waves".

Because h is so small, this exponent is wildly oscillating over any reasonably-sized region of space unless the momentum is also very small. This is the reason we don't see bullets interfere: the interference pattern is far too fine to see. We can only see interference for small particles like electrons, where the small mass means a small momentum per particle.

Some more comments about this important equation:

1. The factor we have ignored by writing "$\propto$" contains some not-very-important numerical constants, and a somewhat-more-important factor of $1/|\vec{r}_1 - \vec{r}_2|$. The latter means that the amplitude decreases as $\vec{r}_1$ and $\vec{r}_2$ get farther apart, which makes sense – waves spread out.

2. Note I said "definite momentum" $\vec{p}$. The only way this is consistent with the uncertainty principle is for the uncertainty in position $\Delta\vec{r}$ to be infinite. In this expression, it is indeed infinite: $\langle\vec{r}_2|\vec{r}_1\rangle$ is non-zero for any r_1 and r_2, no matter how far apart. Thus there is complete uncertainty in position – a free particle can be anywhere. (Those of you who know some relativity may notice this contradicts the fact that information cannot travel faster than the speed of light; to make $\langle\vec{r}_2|\vec{r}_1\rangle$ consistent with causality requires much more work.)

3. This exponential is a solution of the three-dimensional wave equation $e^{i\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}$ as you showed on the first homework, if you set $\vec{p}=\hbar\vec{k}$. In fact, if you solve the wave equation in radial coordinates, you also get the $1/r$ piece in front as well. They really are waves!

The particle in a box, again

Let's redo the particle in a box in our new picture. When the momentum p is in the same direction as $\vec{r}_1-\vec{r}_2$ (as it always must be in one dimension), then we have

$$\langle 1|2\rangle \propto e^{i|\vec{p}|d/\hbar} = e^{i2\pi d/\lambda}$$

where $d=|\vec{r}_1-\vec{r}_2|$ is the distance from a to b.

Now let's specialize this to one dimension. We can label the coordinate as x. Our formula says that if the momentum is exactly p, then the amplitude to get from x_1 to x_2 is

$$\langle x_2|x_1\rangle = e^{ip(x_1-x_2)/\hbar}$$

Note that even though we are in one dimension, p is still a vector: it can be positive (momentum going to the right) or negative (momentum going to the left). Thus there are *two* solutions with the same magnitude momentum.

By definition, a particle in a box has amplitude 0 to be found at the edges. Thus a particle in a box can't have a fixed momentum: $e^{ipx/\hbar}$ is never zero. But in quantum mechanics, a particle doesn't have to have a fixed momentum. Just as the amplitude in the double slit experiment is the sum over different paths, the amplitude can also consist of a sum over different momenta. As I just said, in a box the particle can have positive or negative momentum. So let $\phi_{\text{box}}(x)$ be the probability amplitude to find a particle at a position x inside the box. Let's fix one end of the box to be at x=0, the other to be at $x=L$. We have $\phi(x)=0$ for $x\leq 0$ and $x\geq L$, but for $0\leq x\leq L$ we have

$$\phi_{\text{box}}(x) = A\phi_p + B\phi_{-p}$$

for some coefficients A and B. These two terms correspond to different momenta: one to a particle going to the left, and the other the right.

Now let's find what values of A, B and p have $\phi_{\text{box}}(0) = 0$ and $\phi_{\text{box}}(L) = 0$. The former requires

$$\phi_{\text{box}}(0) = A + B = 0$$

so we must have $A = -B$. The second requires

$$\begin{aligned} 0 &= \phi_{\text{box}}(L) \\ &= Ae^{ipL/\hbar} - Ae^{-ipL/\hbar} \\ &= 2iAsin[pL/\hbar] \end{aligned}$$

This is true if $A = 0$, but that means there is a vanishing amplitude in the entire box. Because $sin(n\pi) = 0$ for any integer n, we need to have

$$p = \frac{n\hbar\pi}{L} = nh2L$$

for the amplitude to vanish at the box edges. Plugging this and $A = -B$ back into the original expression for ϕ_{box} gives

$$\phi_{\text{box}}(x) = 2iA\sin(nx\pi/L) \qquad\qquad 0 \leq x \leq L$$

We can fix A as well, but we'll come back to that later in the class.

The probability amplitude is a standing wave in the box. There are only certain possible wavelentgths $\lambda = 2L/n$ allowed, which translates into the $p = hn/(2L)$ requirement. Thus we see the "wave" nature of particles is in the probability amplitude!