Modern Physics notes – Spring 2006
Paul Fendley fendley@virginia.edu

Lecture 9

- Crystal diffraction
- The double slit for the last time

- Feynman, 2.3, 3.2-3.3

Crystal diffraction

This is the scattering of waves off of crystals. In many materials, the atoms arrange themselves in regular patterns called crystals. Thus we have a source s, which we take to be far away from the crystal. We observe the scattered wave at a point x, which we also take to be fairly far away (you'll see what I mean by "fairly far"). Choose a point x so that it will only observe scattered waves (i.e. it is not in the way of the original wave).

In our new notation, the in state is $|s\rangle$, and the out state is $\langle x|$, just like before. To simplify the problem a little, we don't worry about the precise kind of atom in the crystal: we just say that when the incoming wave hits an atom, there is a probability a it will be scattered. Then the probability amplitude for the scattering off the atoms 1, 2, ... is

$$\begin{aligned}\langle x|s\rangle &= a\langle x|1\rangle\langle 1|s\rangle + a\langle x|2\rangle\langle 2|s\rangle + ... \\ &= a\sum_j \langle x|j\rangle\langle j|s\rangle\end{aligned}$$

So this looks just like an experiment with multiple slits.

To simplify the problem a little bit, let's look at the case where the scattering occurs just at planes. Some materials have strong interactions along planes, but weak interactions with the atoms in the other planes. Graphite (a particular arrangement of carbon) is like this: this is why we use it for pencils. It's easy to break the bonds between planes, which is why the planes of graphite slide off on to your paper (and off it when you erase).

Thus when you scatter off graphite, it's reasonable to consider the case where you consider scattering off planes instead of individual atoms. So let's consider scattering off two adjacent planes in the crystal, and understand when the scattering is constructive or destructive. In our old way of arguing, we would just say that they interfere constructively if the difference in distances they travel is equal to an integer number of wavelengths, destructively if the difference is a half-integer number of wavelengths. For light incident at angle θ to planes a distance d apart, the difference between the two distances is $2d\sin\theta$ (This is where I used the approximation that the detector at x is "fairly far" away; this means that the outgoing waves are parallel.) The condition for constructive interference is therefore

$$2d\sin\theta = n\lambda$$

for n an positive integer. To get destructive interference, replace n in this formula with $(n+1/2)$.

Let's rederive this in our new way of writing things. The probability amplitude for scattering off plane 1 or plane 2 is

$$\langle x|s\rangle = a\langle x|1\rangle\langle 1|s\rangle + a\langle x|2\rangle\langle 2|s\rangle$$

(We assume that $a \ll 1$ so that most of the wave makes it past the first plane, and so that the probability the wave scatters off the second plane is a, the same as for the first.) Destructive interference holds when these two terms cancel each other, so that

$$\frac{\langle x|1\rangle\langle 1|s\rangle}{\langle x|2\rangle\langle 2|s\rangle} = -1$$

For each of these terms, the momentum is in the same direction as the two points. (We needed to break it down this way for this to be true, since the momentum changes as a result of the scattering.) Using the formula above for each of these four amplitudes in the condition for destructive interference gives

$$e^{ipd_{12}/\hbar} = -1$$

where d_{12} is the difference in the total path lengths. This may look different, but it is the same condition as before. Remember that $e^{i\pi} = -1$ and $e^{i2n\pi} = 1$, so $e^{i(2n+1)\pi} = -1$. Thus for destructive interference

$$pd_{12} = (2n+1)\pi\hbar$$

Since the difference $d_2 - d_1$ is still $2d\sin\theta$, and $p = h/\lambda$, we recover for destructive interference

$$2d\sin\theta = (n+1/2)\lambda.$$

For constructive interference, we have

$$e^{ipd_{12}/\hbar} = +1$$

and we recover the correct formula $2d\sin\theta = n\lambda$.

Since this (re-)derivation was a little quick, let's do it in excruciating detail. Let the initial momentum be $\vec{p}_0$ and the final be $\vec{p}_f$. We assume the scattering is elastic (KE is conserved) so that $|\vec{p}_0| = |\vec{p}_f|$, but their directions are of course different. Let $\vec{r}_1$ be the location on the first plane the wave hits, $\vec{r_2}$ be the location on the second plane, $\vec{s}$ be the location of the source, and $\vec{x}$ be the location of the detector. For destructive interference, we then have

$$
\begin{aligned}
-1 &= \frac{\langle x|1\rangle\langle 1|s\rangle}{\langle x|2\rangle\langle 2|s\rangle} \\
&= \frac{e^{i\vec{p}_f\cdot(\vec{r}_1-\vec{x})/\hbar}e^{i\vec{p}_0\cdot(\vec{s}-\vec{r}_1)/\hbar}}{e^{i\vec{p}_f\cdot(\vec{r}_2-\vec{x})/\hbar}e^{i\vec{p}_0\cdot(\vec{s}-\vec{r}_2)/\hbar}} \\
&= e^{i\vec{p}_f\cdot(\vec{r}_1-\vec{r}_2)/\hbar}e^{i\vec{p}_0\cdot(\vec{r}_2-\vec{r}_1)/\hbar} \\
&= e^{2id|\vec{p}|\sin\theta/\hbar} \\
&= e^{4id\sin\theta/\lambda}
\end{aligned}
$$

where I used the fact (easy to see by drawing the picture) that $\vec{p}_0\cdot(\vec{r}_2-\vec{r}_1) = \vec{p}_f\cdot(\vec{r}_1-\vec{r}_2) = |\vec{p}|\sin(\theta)$.

Note that the formula for constructive interference has an important thing in common with the photoelectric effect. For λ large enough, this formula cannot be satisfied for any value of θ. The maximum value of $\sin\theta$ is 1 and the minimum value of n is 1. Thus if $\lambda > 2d$, there is no way of satisfying this condition. Whatever you send in does not scatter!

Why did Feynman choose graphite and neutrons in his example?

More complicated arrangements of atoms cause more complicated interference patterns. A few patterns are given in the book. The key point is to use light or particles of a wavelength of roughly the same size as the features you're trying to understand. The reason is the same as above, if the wavelength is too long, it can't constructively interfere. If it is too short, it can scatter but you won't be able to see the interference pattern. If you're using light, it's therefore common to use X-rays, because their wavelength is about the same size as typical distances between atoms. When using particle scattering to study materials, it is common to use neutrons. Why do they use neutrons instead of protons or electrons?

The double slit for the last time

Our new approach may seem more complicated, but once you get used to it, it makes things much clearer. For example, we can now understand the double-slit experiment more carefully. Before I stated that either "interference happens", or "interference doesn't happen". The situation really isn't that simple – one can in fact smoothly interpolate between the two possibilities.

Let's make our set-up a little more precise. To "see" the electron, we need to have a detector for the photons scattered by the electron. So we put detector D_1 near slit one and detector D_2 by slit 2. So now we must consider the final state of both the photon *and* the electron. So let's consider the final state $\langle D_1, x|$, which means that we observe both a particle at position x

on the backstop, and a photon at detector D_1. Let's compute the amplitude for this situation $\langle D_1, x|s\rangle$. We denote by a the amplitude that the particle goes through slit 1 and scatters a photon into detector D_1. We denote by b the amplitude that the particle go through slit 2 and still hit detector D_1. Now if we've set up the experiment well, then a is much larger than b. However, if we've used light of a wavelength longer than the distance between slits, it is still certainly possible that the particle can go through slit 2 and scatter a photon into detector D_1. This means we have

$$\langle D_1, x|s\rangle = a\langle x|1\rangle\langle 1|s\rangle + b\langle x|2\rangle\langle 2|s\rangle$$

This is just $a\phi_1 + b\phi_2$ in our old notation. So now we can compute the probability to observe a photon at D_1 and an electron at x. It is

$$|\langle D_1, x|s\rangle|^2 = |a\phi_1 + b\phi_2|^2$$

Now say we use short wavelength light and otherwise set up our apparatus so that $b = 0$. This means that we know that if we observe a photon at D_1, the electron went though slit 1. Then

$$|\langle D_1, x|s\rangle|^2 \propto |\phi_1|^2 \qquad \text{for } a = 0$$

This means no interference! This is our old result. Likewise, say we use long-wavelength light, so that we can't really tell which slit the electron goes through. Then $a \sim b$, so

$$|\langle D_1, x|s\rangle|^2 \propto |\phi_1 + \phi_2|^2 \qquad \text{for } a \sim b$$

This means interference, but no knowledge of the slit. The advantage of the current way of writing things is that we now know what happens in between the two limiting cases. As we know with less and less certainty which slit the electron went through, we obtain more and more interference.

The key principle here is: if we don't know how the particles got to where they are observed, we must sum the amplitudes for all possibilities.